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Secondary differential operators

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Abstract. *Let Ч be an arbitrary sustem of partial (non-linear) differential equations. Higher infinitesimal symmetries of Ч may be interpreted as vector fields on the «manifold» Sol Ч of all local solutions of this system. The paper deals with construction of differential operators of arbitrary orders on Sol Ч. These approaches to construction of the theory of these operators, geometric and functional are presented, and their equivalence is proved when Ч is the trivial equation. Coincidence of «extrinsic» and «intrinsic» geometric secondary operators is proved for an arbitrary system Ч. It is shown that each geometric secondary operator may be approximated by a sum of compositions of evolution differentiations with any possible accuracy, a description of geometric secondary operators in local coordinates is also given. These results are obtained by studying the geometry of infinite jets and infinitely prolonged equations.*

INTRODUCTION

It is well known that ordinary differential equations describing classical particles are characteristics for partial differential equations describing corresponding quantum particles. This is the basic line connecting classical and quantum mechanics of particles. In particular, the quantization problem may be viewed as the problem to reconstruct partial differential equation if ordinary equations

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of characteristics are known.

Having this in mind one can expect that classical field theory is similarly connected with the quantum field theory. Assuming this one is forced to postulate the existence of such equations, of genuine equations of quantum fields whose characteristics are described by partial differential equations of classical fields. Thise hypotetical equations, as well as the hypotetical operators in their left-hand sides, may be called secondary ones, or more speculatively, secondary quantized, since partial differential operators may be considered as quantized ordinary operators and in quantum field theory the quantization process starts with partial equations, i.e. quantized equations. Thus our problem is to find the rigorous mathematical notion of secondary differential operators.

In fact, nowadays we know that there exist other secondary notions. For example, higher infinitesimal symmetries of partial differential equations are nothing but secondary vector fields. Similarly, conservation laws for partial differential equations may be interpreted as secondary differentail forms, etc. . In other words, we observe the existence of the next level differential calculus which may be called secondary.

Our interest in the secondary differential calculus arises from the hypothesis that it is the unique natural language for the quantum field theory in the same sense as the classical differential calculus is natural for the classical field theory. Further motivations and general description of the Secondary Calculus the reader may find in [1], [2].

In this paper we answer the question what are secondary differential operators. For simplicity only scalar operators are considered here. Our main tool is the geometry of infinite jet manifolds and infinitely prolonged partial differential equations in the spirit of [3].

The main results of this paper was annunced in [4].

§1. PRELIMINARIES

In this paper we deal with the category of smooth manifolds and smooth maps.

Some necessary notions and results from [5], [3] are given in this section in the convenient form.

1.1. *Jet space.* Let N be a smooth manifold, $\dim N = m + n$. The class of n--dimensional submanifolds $L \subset N$ mutually tangent to each others with the order k tangency at point $x \in N$ will be called the k-jet of an n-dimiensional submanifold at x and denoted $[L]_x^k$.

Set

$$N_m^k(x) = \{[L]_x^k \mid \dim L = n, x \in L\} \text{ and } N_m^k = \bigcup_x N_m^k(x).$$

The set N_m^k has a natural structure of a smooth manifold. For $k \geqslant 1$ there are projections $¶_{k,1} : N_m^k \longrightarrow N_m^1$,

$$¶_{k,1}([L]_x^k) = [L]_x^l.$$

If $L \subset N$ then there is a smooth map

$$j_k(L) : L \to N_m^k, \quad j_k(L)(x) = [L]_x^k.$$

Clearly $¶_{k,1} \circ j_k(L) = j_l(L)$. The inverse limit of the chain of maps

$$\ldots \longrightarrow N_m^k \xrightarrow{¶_{k,k-1}} \ldots \xrightarrow{¶_{1,0}} N_m^0 = N$$

will be denoted by N_m^∞. Denote the limit of the maps $j_k(L)$, $k \to \infty$ by $j_\infty(L)$: $: L \longrightarrow N_m^\infty$ and the natural projection $N_m^\infty \longrightarrow N_m^k$ by $¶_{\infty,k}$.

The set N_m^k, $0 \leqslant k \leqslant \infty$ is called the manifold of k-jets of n-dimensional submanifolds of N. Then we set $F_k(N) = C^\infty(N_m^k)$ and $F(N)$ denotes the direct limit of the chain of maps

$$\ldots \longrightarrow F_{k-1}(N) \xrightarrow{¶^*_{k,k-1}} F_k(N) \longrightarrow \ldots.$$

Let M^n be an n-dimensional manifold and $¶ : E \longrightarrow M^n$ a submersion, $\dim E =$ $= m + n$. Then the set of k-jets of images of local sections of this submersion forms an open set $J^k¶$ in E_m^k (called the manifold of k-jets of the bundle $¶$ if $¶$ is a bundle). The set of local sections of a submersion $¶$ will be denoted by $\Gamma_{\text{loc}}(¶)$ and U_s denotes the domain of s for $s \in \Gamma_{\text{loc}}(¶)$. We set

$$j_k(s) = j_k(L) \circ s,$$

where $L = s(U_s)$, and

$$¶_k = ¶ \circ ¶_{k,0}, \quad F_k(¶) = C^\infty(J^k(¶)), \quad 0 \leqslant k \leqslant \infty, \quad F(¶) = \lim_{k \to \infty} \operatorname{dir} F_k(¶).$$

Let $U = V^n \times U^m$, where V^n (resp. U^m) be a domain in $\mathbb{R}^n$ (resp. in $\mathbb{R}^m$), and (q, p), where $q = (q_1, \ldots, q_n)$, $p = (p^1, \ldots, p^m)$, a corresponding coordinate system. Then on $J^k\alpha$, $0 \leqslant k \leqslant \infty$, where $\alpha : U \longrightarrow V^n$ is the natural projection, the coordinate system arises:

$$q_j, p_\sigma^i$$

$j = 1, 2, \ldots, n$; $i = 1, 2, \ldots, m$; $\sigma = (i_1, \ldots, i_n)$, $i_1, \ldots, i_n = 0, 1, 2, \ldots, |\sigma| =$ $= i_1 + \ldots, + i_n$.

Functions p^i_σ are uniquely determined by the following property

$$j_k(s)^*(p^i_\sigma) = \frac{\partial^{|\sigma|} s^i}{\partial q_1^{i_1} \ldots \partial q_n^{i_n}},$$

where $s \in \Gamma(\alpha)$ is defined by $p^i = s^i(q)$.

Every diffeomorphism $f: U \longrightarrow U' \subset N$ naturally induces diffeomorphisms $f_{(k)}: J^k\alpha \longrightarrow N^k_m$ due to which the above coordinates on $J^k\alpha$ are transferred to im $f_{(k)}$. Below they are called the canonical local coordinates and the domain im $f_{(k)}$ with these coordiantes is called a canonical chart.

Let $\Lambda^i(M)$ denote the $C^\infty(M)$-module of smooth differential i-forms on a manifold M. We set

$$C\Lambda^i(N^k_m) = \{\omega \in \Lambda^i(N^k_m) \mid j_k(L^n)^*\omega = 0 \text{ for any } L^n \subset N\},$$

$$C\Lambda^i(J^k\P) = \{\omega \in \Lambda^i(J^k\P) \mid j_k(s)^*\omega = 0 \text{ for any } s \in \Gamma_{\mathrm{loc}}(\P)\}.$$

Clearly,

$$\P^*_{k,l}(C\Lambda^i(N^l_m)) \subset C\Lambda^i(N^k_m) \text{ and } \P^*_{k,l}(C\Lambda^i(J^l\P)) \subset C\Lambda^i(J^k\P),$$

which allows one to define submodules

$$C\Lambda^i(N^\infty_m) \subset \Lambda^i(N^\infty_m) \text{ and } C\Lambda^i(\P) \subset \Lambda^i(\P),$$

where

$$\Lambda^i(N^\infty_m) = \lim_{k\to\infty} \operatorname{dir} \Lambda^i(N^k_m) \text{ and } \Lambda^i(\P) = \lim_{k\to\infty} \operatorname{dir} \Lambda^i(J^k\P),$$

$$C\Lambda^i(N^\infty_m) = \lim_{k\to\infty} \operatorname{dir} C\Lambda^i(N^k_m) \text{ and } C\Lambda^i(\P) = \lim_{k\to\infty} \operatorname{dir} C\Lambda^i(J^k\P).$$

The submodule $C\Lambda^1(N^k_m) \subset \Lambda^1(N^k_m)$ is of constant rank and dualizing determines the distribution $\theta_k \longrightarrow C_{\theta_k} \subset T_{\theta_k}(N^k_m)$ called the Cartan distribution. If $\theta \in N^\infty_m$ and $\theta = \{\theta_k\}$, where $\theta_k \in N_m$, $\P_{k,1}(\theta_k) = \theta_l$, then we determine the tangent space $T_\theta(N^\infty_m)$ as the inverse limit of the chain of linear maps

$$\ldots \longrightarrow T_{\theta_k}(N^k_m) \xrightarrow{d\P_{k,k-1}} T_{\theta_{k-1}}(N^{k-1}_m) \longrightarrow \ldots.$$

Since $d\P_{k,k-1}(C_{\theta_k}) \subset C_{\theta_{k-1}}$, there is defined the inverse limit, C_θ, of the chain

$$\ldots \longrightarrow C_{\theta_k} \xrightarrow{d\P_{k,k-1}} C_{\theta_{k-1}} \longrightarrow \ldots.$$

The distribution $C(N): \theta \longrightarrow C_\theta$ is called the Cartan distribution on N^∞_m. The module $C\Lambda^1(N^\infty_m)$ annihilating the Cartan distribution within the canonical chart is generated by the forms

$$U(p^i_\sigma) = \mathrm{d}p^i_\sigma - \sum_{j=1}^{n} p^i_{\sigma+1_j}\mathrm{d}q_j,$$

where $\sigma + 1_j = (i_1, \ldots, i_{j-1}, i_j + 1, i_{j+1}, \ldots, i_n), |\sigma| \leqslant k$.

The maximal integral manifolds of the Cartan distribution N^∞_m is of dimension n and locally is of the form $\operatorname{im} j_\infty(L)$.

1.2. *Differential equations.* In what follows a systems of non-linear partial differntial equations is considered as a closed submanifold in N^k_m (in $J^k\P$). Here n is the number of independent variables m the number of dependent variables, k the order of the system. Instead of «the system of equations» we will just say «equation».

Let $\text{Ч} \subset N^k_m$ $(J^k\P)$ be an equation. A submanifold $L^n \subset N$ $(s \subset \Gamma_{\mathrm{loc}}(\P))$ is called its solutions if $\operatorname{im} j_k(L) \subset \text{Ч}$ $(\operatorname{im} j_k s \subset \text{Ч})$. The set of solutions is denoted by $\operatorname{Sol}\text{Ч}$.

Define the i-th prolongation $\text{Ч}_i \subset N^{k+i}_m$ $(\text{Ч}_i \subset J^{k+i}\P)$ of the equation $\text{Ч} \subset N^k_m$ $(\text{Ч} \subset J^k\P)$ assuming that $[L]^{k+i}_x \subset \text{Ч}_i$ if and only if $\operatorname{im} j_k(L)$ is tangent to Ч at $j_k(L)(x)$ with tangency of order i. Then $\P_{i,r}(\text{Ч}_i) \subset \text{Ч}_r$ for $i \geqslant r$. The inverse limit of the system

$$\ldots \longleftarrow \text{Ч}_i \xleftarrow{\;\P_{i+1,i}|\text{Ч}_{i+1}\;} \text{Ч}_{i+1} \longleftarrow \ldots$$

is denoted by Ч_∞.

The equation Ч called a formally integrable equation, if each its prolongation Ч_i is a smooth submanifold in N^{k+i}_m (in $J^{k+i}\P$) and projections $\P_{k+i+1,k+i}|\text{Ч}_{i+1} : \text{Ч}_{i+1} \longrightarrow \text{Ч}_i$ are vector bundles. Below we only deal with formally integrable equations.

We set

$$F(\text{Ч}) = F(N)\,|\,\text{Ч}_\infty(F(\P)\,|\,\text{Ч}_\infty), \quad \Lambda^q(\text{Ч}) = \Lambda^q(N^\infty_m)\,|\,\text{Ч}_\infty \quad (= \Lambda^q(\P)\,|\,\text{Ч}_\infty)$$

and

$$C\Lambda^q(\text{Ч}) = C\Lambda^q(N^\infty_m)\,|\,\text{Ч}_\infty \quad (= C\Lambda^q(\P)\,|\,\text{Ч}_\infty).$$

In case $\text{Ч} \subset J^k\P$ the subalgebra $(\P_\infty\,|\,\text{Ч}_\infty)^*\,(C^\infty(M))$ in $F(\text{Ч})$ is identified with $C^\infty(M)$, and the subspace $(\P_\infty\,|\,\text{Ч}_\infty)^*\,(\Lambda^q(M))$ in $\Lambda^q(\text{Ч})$ with $\Lambda^q(M)$. The restriction of the Cartan distribution on Ч_∞ is denoted by $C(\text{Ч})$.

1.3. *Symmetries of differential equations.* Denote by $D(F(\text{Ч}))$ the set of all differentiations of the algebra $F(\text{Ч})$ (i.e. the set of all vector on Ч_∞). The Lie derivative of the form $\omega \in \Lambda^q(\text{Ч})$ along the vector field X is denoted by $X(\omega)$.

We set

$$D_{\mathcal{C}}(F(\text{Ч})) = \{X \in D(F(\text{Ч})) \mid X(\mathcal{C}\Lambda^1(\text{Ч})) \subset \mathcal{C}\Lambda^1(\text{Ч})\},$$

$$\mathcal{C}D(F(\text{Ч})) = \{X \in D(F(\text{Ч})) \mid X \rfloor \mathcal{C}\Lambda^1(\text{Ч}) = 0\}.$$

The fields $X \in D_{\mathcal{C}}(F(\text{Ч}))$ are called $\mathcal{C}$-fields and the fields $X \in \mathcal{C}D(F(\text{Ч}))$ are called trivial $\mathcal{C}$-fields.

We need the following properties of $\mathcal{C}$-fields (see [3], [5]):

1) $D_{\mathcal{C}}(F(\text{Ч}))$ is a subalgebra in the Lie algebra $D(F(\text{Ч}))$;

2) $\mathcal{C}D(F(\text{Ч}))$ is the ideal in the Lie algebra $D_{\mathcal{C}}(F(\text{Ч}))$;

3) $X \in D_{\mathcal{C}}(F(\text{Ч}))$ if and only if $[X, \mathcal{C}D(F(\text{Ч}))] \subset \mathcal{C}D(F(\text{Ч}))$.

The quotient algebra $\operatorname{Sym} \text{Ч} = D_{\mathcal{C}}(F(\text{Ч}))/\mathcal{C}D(F(\text{Ч}))$ is called the algebra of intrinsic infinitesimal symmetries of Ч.

If $\text{Ч}_\infty = N_m^\infty$ $(= J^\infty \P)$ then we write $\varkappa(N)$ $(\varkappa(\P))$ instead of $\operatorname{Sym} \text{Ч}$.

4) Any $\mathcal{C}$-field $X \in D_{\mathcal{C}}(F(\text{Ч}))$ is a restriction on Ч_∞ of a $\mathcal{C}$-field $Y \in D_{\mathcal{C}}(F(N))$;

5) $X \in \mathcal{C}D(F(\text{Ч}))$ if and only if a $\mathcal{C}$-field $Y \in D_{\mathcal{C}}(F(N))$ such that $X = Y \mid \text{Ч}_\infty$ can be restricted on $\operatorname{im} J_\infty(L^n)$ for any $L^n \subset N$.

Given a bundle $\P : E \longrightarrow M^n$ any vector field X on M^n determines the vector field $\hat{X} \in \mathcal{C}D(F(\text{Ч}))$ by the formula

$$\hat{X}(f)(j_\infty(s)(x)) = X(j_\infty(s)^* f)(x),$$

where $f \in F(\P)$, $s \in \Gamma_{\text{loc}}(\P)$, $x \in M$.

Let q_j, p_σ^i be canonical local coordinates on $J^\infty \P$. Then

$$\frac{\hat{\partial}}{\partial q_j} = \frac{\partial}{\partial q_j} + \sum_{i,\sigma} p^i_{\sigma + 1_j} \frac{\partial}{\partial p^i_\sigma}.$$

Denote the operators $\dfrac{\hat{\partial}}{\partial q_j}$ by D_j.

Put $D(M)$ for the set of all the vector fields on M.

If $X = \sum_{i=1}^{n} f_i \dfrac{\partial}{\partial q_i} \in D(M)$, then $\hat{X} = \sum_{i=1}^{n} f_i \cdot D_i$.

1.4. *$\mathcal{C}$-differential operators*. The algebra $F(\text{Ч})$ is naturally filtered by subalgebras $F_i(\text{Ч})$, where $F_i(\text{Ч})$ is the image of $C^\infty(\text{Ч}_{i-k})$ under $\P^*_{\infty,i}$ (here k is the order of Ч). An $\mathbb{R}$-linear map $\Delta : F(\text{Ч}) \longrightarrow F(\text{Ч})$ is called a linear differential operator of order $\leqslant l$ if

1) $\Delta(F_i(\text{Ч})) \subset F_{i+j}(\text{Ч})$, $j = j(i)$;

2) $\delta_{f_0}(\ldots(\delta_{f_1}(\Delta)\ldots) = 0$ for any $f_0, \ldots, f_1 \in F(\text{Ч})$, where $\delta_f(\Delta) = [\Delta, f]$. The details see in [3].

Differential operators from a filtered $F(\text{Ч})$-module P into a filtered $f(\text{Ч})$-

-module Q are similarly defined.

The set of all linear differential operators (of order $\leqslant l$) from $F(\text{ч})$ into itself is denoted by $\mathrm{Diff}\,(F(\text{ч}))$ $(\mathrm{Diff}_l\,(F(\text{ч})))$.

An operator $\Delta \in \mathrm{Diff}\,(F(N))$ is a $\mathcal{C}$-differential operator if for any submanifold $L^n \subset N$ the operator Δ admits the restriction onto the submanifold $\mathrm{im}\, j_\infty(L^n) \subset \subset N^\infty_m$.

An operator $\Delta \in \mathrm{Diff}\,(F(\text{ч}))$ is a $\mathcal{C}$-differential operator if there is a $\mathcal{C}$-differential operator $\Delta' \in \mathrm{Diff}\,(F(N))$ such that $\Delta = \Delta' \mid \text{ч}_\infty$.

The $F(\text{ч})$ algebra of $\mathcal{C}$-differential operators is denoted by $\mathcal{C}\,\mathrm{Diff}\,(F(\text{ч}))$.

The following statements hold (see [5]):

1) The algebra of $\mathcal{C}$-differential operators $\mathcal{C}\,\mathrm{Diff}\,(F(\text{ч}))$ is generated by $\mathcal{C}$-differential operators of order $\leqslant 1$

2) If $\text{ч} \subset J^k \P$, then the algebra $\mathcal{C}\,\mathrm{Diff}\,(F(\text{ч}))$ is generated by $\mathcal{C}$-differential operators of the form $\hat{X} \mid \text{ч}_\infty$, where $X \in D(M)$, and functions $\varphi \in F(\text{ч})$.

The most important example of $\mathcal{C}$-differential operators are those of universal linearization (see [5]). Recall that if P is an $F(\P)$-module of smooth sections of a finite-dimensional vector bundle over $J^\infty \P$ and $p \in P$, then in canonical coordinates q_j, p^i_σ the corresponding universal linearization operator l_p is of the form

$$l_p = \begin{pmatrix} \sum\limits_\sigma \dfrac{\partial F_1}{\partial p^1_\sigma} \circ D_\sigma & \ldots & \sum\limits_\sigma \dfrac{\partial F_1}{\partial p^m_\sigma} \circ D_\sigma \\ \ldots & \ldots & \ldots \\ \sum\limits_\sigma \dfrac{\partial F_r}{\partial p^1_\sigma} \circ D_\sigma & \ldots & \sum\limits_\sigma \dfrac{\partial F_r}{\partial p^m_\sigma} \circ D_\sigma \end{pmatrix},$$

where $p = (F_1, \ldots, F_r)$, $F_i \in F(\P)$; $D_\sigma = D_1^{i_1} \circ \ldots \circ D_n^{i_n}$, $\sigma = (i_1, \ldots, i_n)$.

Note that $l_p \equiv 0$ if and only if $F_i = F_i(q_1, \ldots, q_n) \in C^\infty(M^n)$, $i = 1, 2, \ldots, r$.

Besides,

$$l_{fp} = f \cdot l_p + p \cdot l_f,$$

where $p \in P$, $f \in F(\P)$.

1.5. *Local regular differential equations*. Denote by $I(\text{ч}_\infty)$ the ideal in $F(N)$ considering of functions vanishing on ч_∞. Let the equation ч in a neighbourhood $V \subset N^k_m$ be determined by a collection of functions $\{\mathcal{S}_\nu\}\nu = 1, 2, \ldots, R$, i.e.

$$\text{ч} \cap V = \{\theta_k \in N^k_m \mid \mathcal{S}_\nu(\theta_k) = 0, \nu = 1, 2, \ldots, R\}.$$

Then the functions of $\mathcal{C}\,\mathrm{Diff}\,(\mathcal{S}_\nu)$, $\nu = 1, 2, \ldots, R$ belong to $I(\text{ч}_\infty)$ in the neighnourhood $\P^{-1}_{\infty,k}(V)$. If in this neighbourhood $I(\text{ч}_\infty) = \sum\limits_\nu \mathcal{C}\,\mathrm{Diff}\,(\mathcal{S}_\nu)$ then we say that $\mathcal{Y}$ is regular in V. If for each point $\theta_k \in \mathcal{Y}$ there exists its neighbourhood $V_{\theta_k} \subset N_m^k$ in which $\mathcal{Y}$ is regular, then we say that $\mathcal{Y}$ is locally regular.

We will need the following standard results of formal theory (see e.g. [6]).

Let equation $\mathcal{Y}$ be formally integrable and locally regular and $W' \subset N_m^\infty$ a canonical chart with coordinates q_j, p_σ^i in a neighbourhood of a point from $\mathcal{Y}_\infty$. Then:

1) There is a domain $W \subset W'$ such that coordinates q_j together with some of coordinates p_σ^i (we denote them by $p_{\bar\sigma}^i$ the remaining ones by $p_{\underline{\sigma}}^i$) form a coordinate system on $W \cap \mathcal{Y}_\infty$;

2) Restrictions $\bar p_\sigma^i$ of coordinate functions p_σ^i onto $\mathcal{Y}_\infty$ are functions in $\bar q_j, \bar p_{\bar\sigma}^i$, i.e. $\bar p_\sigma^i = f_\sigma^i(\bar q_j, \bar p_{\bar\sigma}^i)$, and the prolongation $\mathcal{Y}_\infty$ is determined by equations

$$p_{\underline{\sigma}}^i - f_{\underline{\sigma}}^i(q_j, p_{\bar\sigma}^i) = 0.$$

In this work we only consider formally integrable and locally regular differential equations.

Everywhere in this work we denote the restriction of a function or an operator onto $\mathcal{Y}_\infty$ barring the symbol of this function or operator, e.g. $\bar f = f \,|\, \mathcal{Y}_\infty$, $\bar D_j = D_j \,|\, \mathcal{Y}_\infty$. The summation sign for repeated indices and multi-indicies in cumbersome formulas will be often omitted.

We will need the following technical results.

Let $\mathcal{Y} \subset J^k \P$. An operator $\Delta \in \mathrm{Diff}\,(F(\mathcal{Y}))$ is called vertical if

$$[\Delta, f] = 0 \quad \text{for any} \quad f \in C^\infty(M).$$

LEMMA 1.5.1. *Each operator $\Delta \in \mathrm{Diff}\,(F(\mathcal{Y}))$ in canonical coordinates $\bar q_j$, $\bar p_{\bar\sigma}^i$ on $W \cap \mathcal{Y}_\infty$ is uniquely represented in the form*

$$\Delta = \sum_\sigma \nabla_\sigma \circ \bar D_\sigma ,$$

where ∇_σ are vertical operators and $\bar D_\sigma = \bar D_1^{i_1} \ldots \bar D_n^{i_n}$, $\sigma = (i_1, \ldots, i_n)$.

Proof. In coordinates $\bar q_j$, $\bar p_{\bar\sigma}^i$ on $W \cap \mathcal{Y}_\infty$ an operator $\Delta \in \mathrm{Diff}\,(F(\mathcal{Y}))$ is expressed in the form

$$\Delta = \sum_{s+|\tau|=0} \sum_{\substack{i_1,\ldots,i_s \\ \sigma_1,\ldots,\sigma_s,\tau}} a_{\bar\sigma_1,\ldots,\bar\sigma_s\tau}^{i_1,\ldots,i_s} \frac{\partial^{s+|\tau|}}{\partial \bar p_{\bar\sigma_1}^{i_1} \ldots \partial \bar p_{\bar\sigma_s}^{i_s} \partial \bar q_\tau} ,$$

where

$$\frac{\partial^{|\tau|}}{\partial \bar{q}_\tau} = \left(\frac{\partial}{\partial \bar{q}_1}\right)^{j_1} \circ \ldots \circ \left(\frac{\partial}{\partial \bar{q}_n}\right)^{j_n}, \quad \tau = (j_1, \ldots, j_n).$$

If in this formula we substitute operators $\left(D_j - \sum_{i,\sigma} p^i_{\sigma+1_j} \frac{\partial}{\partial p^i_\sigma}\right) | Ч_\infty$ for $\frac{\partial}{\partial \bar{q}_j}$ then Δ may be reduced to the form $\Delta = \sum_\sigma \nabla_\sigma \circ \bar{D}_\sigma$, where operators ∇_σ contain the derivatives only with respect to variables $\bar{p}^i_\sigma$, and therefore commute with all functions $f \in C^\infty(M)$.

Now we will prove that this representation is the only possible one. Let $0 = \sum_\sigma \nabla_\sigma \circ \bar{D}_\sigma = \sum_{|\sigma|=k} \nabla_\sigma \cdot \bar{D}_\sigma +$ terms of order $< k$ in $\bar{D}_j$. Consider the operator

$$\delta_{\bar{q}_\tau} = (\delta_{\bar{q}_1})^{i_1} \ldots (\delta_{\bar{q}_n})^{i_n}$$

where $= (i_1, \ldots, i_n), |\tau| = k$.

Obviously, for any multiindex $\sigma, |\sigma| \leqslant k$:

$$\delta_{\bar{q}_\tau}(\bar{D}_\sigma) = \begin{cases} \tau! & \text{if} \quad \sigma = \tau \\ 0 & \text{if} \quad \sigma \neq \tau, \end{cases}$$

where $\tau!$ is the product of factorials of all indices from multiindex τ. Then

$$0 = \delta_{\bar{q}_\tau}(0) =$$

$$= \delta_{\bar{q}_\tau}\left(\sum_{|\sigma|=k} \nabla_\sigma \circ \bar{D}_\sigma + \text{terms of order} < k \text{ in } \bar{D}_j\right) =$$

$$= \sum_\sigma \delta_{\bar{q}_\tau}(\nabla_\sigma \circ \bar{D}_\sigma) = \sum_\sigma \nabla_\sigma \circ \delta_{\bar{q}_\tau}(\bar{D}_\sigma) = \tau! \, \nabla_\tau.$$

Hence, $\nabla_\tau = 0, |\tau| = k$, etc. . ■

COROLLARY 1. *Each operator* $\Delta \in \mathrm{Diff}\,(F(Ч))$ *may be uniquely represented as the sum of two operators* $\Delta = \Box + \nabla$, *where* $\Box$ *is a vertical operator, and* $\nabla = \sum_i \Box_i \circ \nabla_i$, *where* $\nabla_i \in \mathcal{C}\,\mathrm{Diff}\,(F(Ч)$ $\nabla_i(1) = 0$ *and* $\Box_i$ *are vertical operators.*

Proof. Proof consists in applying the standard technique of the partition of the unit and the statement of Lemma. ■

COROLLARY 2. *Let* $\Delta = \sum_i \Box_i \circ \nabla_i \in \mathrm{Diff}\,(F(Ч))$, *where* $\nabla_i \in \mathcal{C}\,\mathrm{Diff}\,(F(Ч))$,

$\nabla_i(1) = 0$ *and operators* $\square_i$ *are vertical. If* Δ *is a vertical operator, then* $\Delta = 0$.

Proof. Express the operator Δ in canonical coordinates $\bar{q}_j$, $\bar{p}^i_{\bar{\sigma}}$ on $W \cap Ч_\infty$ in the form

$$\Delta = \sum_{|\sigma|>0} \nabla_\sigma \circ \bar{D}_\sigma,$$

where ∇_σ are vertical operators. In what follows, $0 = [\Delta, \bar{q}_j] = \sum_{\sigma'} \nabla_{\sigma'} \circ \bar{D}_{\sigma'-1_j}$, where the j-th index of multiindices is $\geqslant 1$. By the uniqueness of representation $\nabla_{\sigma'} = 0$ in lemma for all $\nabla_{\sigma'}$. Since it is true for all $j = 1, 2, \ldots, n$, then $\nabla_\sigma = 0$ for all ∇_σ. ∎

REMARK. The same arguments as in the proof of Lemma show that each $\Delta \in$ $\in \mathrm{Diff}\,(F(Ч))$ in canonical coordinates on $W \cap Ч_\infty$ is uniquely represented in the form

$$\Delta = \sum_\sigma \bar{D}_\sigma \circ \nabla_\sigma,$$

where ∇_σ are vertical operators.

1.6. *$\bar{d}$-cohomology and Euler operator.* Let $\bar{\Lambda}^q(Ч) = \Lambda^q(Ч)/C\Lambda^q(Ч)$ and $\bar{\omega} =$ $= \omega + C\Lambda^q(Ч)$, for any $\omega \in \Lambda^q(Ч)$. There is a natural decomposition (see [5])

$$\Lambda^q(Ч) = \Lambda^q_0(Ч) \oplus C\Lambda^q(Ч).$$

Forms from $\Lambda^q_0(Ч)$ are horizontal.

The obvious inclusion $d(C\Lambda^q(Ч)) \subset C\Lambda^{q+1}(Ч)$ allows us to determine the differential $\bar{d} : \bar{\Lambda}^q(Ч) \longrightarrow \bar{\Lambda}^{q+1}(Ч)$. The cohomology of the arising complex

$$0 \longrightarrow F(Ч) \xrightarrow{\bar{d}} \bar{\Lambda}^1(Ч) \xrightarrow{\bar{d}} \ldots \xrightarrow{\bar{d}} \bar{\Lambda}^{n-1}(Ч) \xrightarrow{\bar{d}} \bar{\Lambda}^n(Ч) \longrightarrow 0$$

is denoted by $\bar{H}^q(Ч)$.

If $Ч_\infty = N^\infty_m(J^\infty ¶)$ then instead of $\bar{\Lambda}^q(Ч)$ and $\bar{H}^q(Ч)$ we will write $\bar{\Lambda}^q(N)$ ($\bar{\Lambda}^q(¶)$) and $\bar{H}^q(N)$ ($\bar{H}^q(¶)$) respectively.

On C-differential operators there is defined a conjugation denoted by asterisk. Namely, if $\Delta : P \longrightarrow Q$, then $\Delta^* : \hat{Q} \longrightarrow \hat{P}$, where for an $F(Ч)$-module S we set (see [5])

$$\hat{S} = \mathrm{Hom}_{F(Ч)}(S, \bar{\Lambda}^n(Ч)).$$

If $\bar{\omega} \in \bar{\Lambda}^n(Ч)$, then $l_{\bar{\omega}} : \varkappa(¶) \longrightarrow \bar{\Lambda}^n(¶)$ and $l^*_{\bar{\omega}} : F(¶) \longrightarrow \widehat{\varkappa(¶)}$ for $\widehat{\bar{\Lambda}^n(¶)} = F(¶)$.

Elements of $\overline{\Lambda}^n(\P)$ are interpreted as Lagrangian densities (see [5]). Then the classical Euler operator $\mathscr{E}$ recovering the Euler-Lagrange equations from Lagrangians can be presented in the form

$$\overline{\omega} \xrightarrow{\mathscr{E}} l^*_{\overline{\omega}}(1) = \mathscr{E}(\overline{\omega}),$$

i.e. $l^*_{\overline{\omega}}(1) = 0$ is the Euler-Lagrange equation corresponding to the Lagrangian density $\overline{\omega}$.

In §3 we will need the following two facts which immediately follow from the properties of the universal linearization operator described above:

1) $l^*_{f\overline{\omega}}(1) = l^*_{\overline{\omega}}(f)$ for $f \in C^\infty(M)$;
2) If $\overline{\omega} = dq_1 \wedge \ldots \wedge dq_n \pmod{C\Lambda^n(\P)}$, then $l^*_{f\overline{\omega}} = 0$ if and only if $f \in C^\infty(M)$.

§2. SECONDARY DIFFERENTIAL OPERATORS. GEOMETRIC APPROACH

2.1. *Guiding considerations.* Scalar secondary differential operators in their simplest form should be operators acting on «smooth» functions determined on the «manifold» Sol Ч of local solutions of the system of (non-linear) differential equations Ч compatiable with the localisation operators. The latter means that if $\square$ is a secondary operator, then $\square(f)\,|\,U = \square(f\,|\,U)$, where f is a «smooth» function on the «manifold» Sol Ч. However it would be unrealistic to try to determine a form of secondary operators when dealing directly with «manifolds» of Sol Ч kind trying e.g. to determine the topology, C^∞-structure, etc on them. Instead we will consider the virtual bundle

$$(2.1.1) \qquad Ч_\infty \ldots \longrightarrow \mathrm{Sol}\,Ч.$$

Here we mean the following. The Cartan distribution on is completely integrable and its integral manifolds are nothing but the local solutions of Ч. If the Frobenius theorems were true in the considered infinite dimensional case, then through any point of $Ч_\infty$ the unique solution of Ч would pass, hence $Ч_\infty$ would be stratified into solutions. In other words, the «manifold» Sol Ч might be identified with the «manifold» of integral manifolds of the Cartan distribution on $Ч_\infty$. Therefore Sol Ч might be locally represented as the base of the bundle $Ч_\infty \longrightarrow \mathrm{Sol}\,Ч$. However, the Frobenius theorem on $Ч_\infty$ is not in general true since the solution of the system near the given point is not uniquely determined by values of all its partial derivatives at this point. Therefore in general the bundle $Ч_\infty \longrightarrow \mathrm{Sol}\,Ч$ does not exist. When we say «virtual» we mean that when necessary we will discuss as if it existed. In particular the proposed geometric approach to determining the secondary operators is based on the following considerations.

Let $\eta : M_1 \longrightarrow M_2$ be a smooth bundle. Describe operators on M_2 in terms of some objects determined on M_1. If we apply the obtained description to the virtual bundle (2.1.1) we get a definition of secondary differential operators.

Denote by $FC(\eta)$ the linear space of operators from $\mathrm{Diff}\,(C^\infty(M_1))$ mapping the subspace of functions constant on any fibre of η into itself.

Let $FC'(\eta)$ be a subspace of $FC(\eta)$ consisting of operators mapping the functions constant on any fibre of η to zero. Then the following obvious statement holds, its proof is omitted here.

PROPOSITION 2.1.1. $FC(\eta)/FC'(\eta) = \mathrm{Diff}\,(C^\infty(M_2))$.

The structure of spaces $FC(\eta)$ and $FC'(\eta)$ is described by the following elementary statement subject to a direct verification in coordinates.

PROPOSITION 2.1.2. 1) $\Delta \in FC'(\eta)$ *if and only if* Δ *can be represented in the form* $\Delta = \sum_i \Delta_i \circ \nabla_i$, *where* ∇_i *are vertical vector fields on* M_1 *with respect to* η *and* $\Delta_i \in \mathrm{Diff}\,(C^\infty(M_1))$;

2) $\Delta \in FC(\eta)$ *if and only if* $FC'(\eta) \circ \Delta \subset FC'(\eta)$.

Now assume that we are given a foliation F on the manifold M or, which is the same, a completely integrable distribution on M. Then differential operators on the «manifold» of leaves of this foliation can be defined by using Propositions 2.1.1 and 2.1.2. Denote by $C_F^\infty(U)$ the set of all smooth functions on M constant on the intersection of any leaf of F with a domain $U \subset M$. Further, set

$$FC(F) = \{\Delta \in \mathrm{Diff}\,(C^\infty(M) \mid \ \Delta(f) \mid U \in C_F^\infty(U) \quad \text{if} \quad f \in C_F^\infty(U)\ \forall U \subset M\},$$

$$FC'(F) = \{\Delta \in \mathrm{Diff}\,(C^\infty(M) \mid \ \Delta(f) \mid U = 0 \quad \text{if} \quad f \in C_F^\infty(U)\ \forall U \subset M\}.$$

Obviously $FC(F)$ and $FC'(F)$ are subspaces of the $\mathbb{R}$-linear space $\mathrm{Diff}\,(C^\infty(M))$ and $FC'(F) \subset FC(F)$. Now, taking Proposition 2.1.1 into account the set of all scalar differential operators $\mathrm{Diff}\,C^\infty(\Gamma)$ on the «manifold» of all local leaves of F can be naturally defined by the «formula»

$$\mathrm{Diff}\,C^\infty(\Gamma) = FC(F)/FC'(F).$$

REMARK. We underline here, that we do not try make any sense of the symbol $C^\infty(\Gamma)$.

In this more general situation the following analogye of Propostion 2.1.2 holds.

PROPOSITION 2.1.2.′ 1) *$\Delta \in FC'(F)$ if and only if Δ can be represented in the form $\Delta = \sum_i \Delta_i \circ \square_i$, where $\square_i$ are vector fields tangent to leaves of F and $\Delta_i \in$* $\in \mathrm{Diff}\,(C^\infty(M))$;

2) *$\Delta \in FC(F)$ if and only if $FC'(F) \circ \Delta \subset FC'(F)$.*

Proof. Proposition 2.1.1 implies Proposition 2.1.2′ because definition of spaces $FC(F)$ and $FC'(F)$ have local character and locally each foliation is a bundle. ■

2.2. *Intrinsic secondary operators*. Now it is natural to set

$$FC'(C(\text{Ч})) = \mathrm{Diff}\,(F(\text{Ч})) \circ CD(\text{Ч})$$

$$FC(C(\text{Ч})) = \{\Delta \in \mathrm{Diff}\,(F(\text{Ч})) \mid FC'(C(\text{Ч})) \circ \Delta \subset FC'(C(\text{Ч}))\},$$

following the lines of Proposition 2.1.2′ and taking into account that the Cartan distribution on Ч_∞ determines a virtual bundle, more exactly the foliation $C(\text{Ч})$. Clearly, $FC'(C(\text{Ч}))$ and $FC(C(\text{Ч}))$ are subspaces of the $\mathbb{R}$-linear space $\mathrm{Diff}\,(F(\text{Ч}))$ and $FC'(C(\text{Ч})) \subset FC(C(\text{Ч}))$. The following proposition is the direct corollary of definitions.

PROPOSITION 2.2.1. 1) *$FC'(C(\text{Ч}))$ is a left ideal in the $\mathbb{R}$-algebra* $\mathrm{Diff}\,(F(\text{Ч}))$;

2) *$FC(C(\text{Ч}))$ is a subalgebra in the $\mathbb{R}$-algebra* $\mathrm{Diff}\,(F(\text{Ч}))$;

3) *$FC'(C(\text{Ч}))$ is a two-sided ideal in $FC(C(\text{Ч}))$.*

This proposition allows to define the quotient algebra

$$\text{Диф}\,(F(\text{Ч})) = FC(C(\text{Ч}))/FC'(C(\text{Ч})).$$

DEFINITION. Elements of $\text{Диф}\,(F(\text{Ч}))$ are called secondary (intrinsic) differential operators of the equation Ч.

Further, set

$$FC_k(C(\text{Ч})) = FC(C(\text{Ч})) \cap \mathrm{Diff}_k(F(\text{Ч}))$$

$$FC'_k(C(\text{Ч})) = FC'(C(\text{Ч})) \cap \mathrm{Diff}_k(F(\text{Ч})).$$

Clearly $FC_k(C(\text{Ч}))$ and $FC'_k(C(\text{Ч}))$ are subspaces of the $\mathbb{R}$-linear space $\mathrm{Diff}_k(F(\text{Ч}))$ and $FC'_k(C(\text{Ч})) \subset FC_k(C(\text{Ч}))$. The quotient space

$$\text{Диф}_k(F(\text{Ч})) = FC_k(C(\text{Ч}))/FC'_k(C(\text{Ч}))$$

is naturally realized as a subspace in $\text{Диф}\,(F(\text{Ч}))$. Its elements are called secondary (intrinsic) operators of order $\leqslant k$. Obviously there is a chain of inclusions

$$\text{Диф}_0(F(\text{Ч})) \subset \text{Диф}_1(F(\text{Ч})) \subset \ldots \subset \text{Диф}_k(F(\text{Ч})) \subset \ldots \subset \text{Диф}(F(\text{Ч})).$$

If $\text{Ч}_\infty = N_m^\infty(J^\infty \P)$, then we will write $FC(\mathcal{C}(N))$ $(FC(\mathcal{C}(\P)))$, $\text{Диф}(F(N))$ $(\text{Диф}(F(\P)))$ etc. instead of $FC(\mathcal{C}(\text{Ч}))$, $\text{Диф}(F(\text{Ч}))$ etc..

PROPOSITION 2.2.2. 1) $\text{Диф}_0(F(\text{Ч})) = FC_0(\mathcal{C}(\text{Ч})) = \{ f \in F(\text{Ч}) \mid X(f) = 0$ *for any* $X \in \mathcal{C}D(F(\text{Ч}))\}$;

2) $FC_0(\mathcal{C}(N)) = \mathbb{R}$;

3) $\text{Диф}_1(F(\text{Ч})) = FC_0(\mathcal{C}(\text{Ч})) \oplus \text{Sym}\,\text{Ч}$.

Proof. 1) The obvious identity $FC_0'(\mathcal{C}(\text{Ч})) = 0$ implies $\text{Диф}_0(F(\text{Ч})) = FC_0(\mathcal{C}(\text{Ч}))$.

2) Let $f \in FC_0(\mathcal{C}(\text{Ч}))$, where $f \in F(\text{Ч})$. Then $X \circ f \in FC'(\mathcal{C}(\text{Ч}))$, where $X \in \mathcal{C}D(F(\text{Ч}))$. But $X \circ f = f \circ X + X(f)$. Since $f \circ X \in FC'(\mathcal{C}(\text{Ч}))$, then $X(f) \in FC'(\mathcal{C}(\text{Ч}))$ and therefore $X(f) = 0$. If $\text{Ч}_\infty = N_m^\infty$ it means that $f = \text{const}$.

3) We observe, that $FC_1'(\mathcal{C}(\text{Ч})) = \mathcal{C}D(F(\text{Ч}))$. On the other hand the inclusion $FC'(\mathcal{C}(\text{Ч})) \circ \Delta \subset FC'(\mathcal{C}(\text{Ч}))$ is equivalent to the inclusion $[FC'(\mathcal{C}(\text{Ч})), \Delta] \subset FC'(\mathcal{C}(\text{Ч}))$ because we always have $\Delta \circ FC'(\mathcal{C}(\text{Ч})) \subset FC'(\mathcal{C}(\text{Ч}))$. This implies

$$[\mathcal{C}D(F(\text{Ч}), \Delta] \subset FC_1'(\mathcal{C}(\text{Ч})) = \mathcal{C}D(F(\text{Ч})),$$

if $\Delta \in FC_1(\mathcal{C}(\text{Ч}))$. Further, the operator Δ is uniquely represented in the form $\Delta = \Delta(1) + (\Delta - \Delta(1))$. Then for any $X \in \mathcal{C}D(F(\text{Ч}))$:

$$[X, \Delta] = [X, \Delta(1)] + [X, \Delta - \Delta(1)] = X(\Delta(1)) + \\ + [X, \Delta - \Delta(1)] \in \mathcal{C}D(F(\text{Ч})).$$

Hence

$$X(\Delta(1)) = 0 \quad \text{and} \quad [X, \Delta - \Delta(1)] \in \mathcal{C}D(F(\text{Ч}))$$

$$\text{for any} \quad X \in \mathcal{C}D(F(\text{Ч})).$$

But this means that $\Delta(1) \in FC_0(\mathcal{C}(\text{Ч}))$, $\Delta - \Delta(1) \in D_{\mathcal{C}}(F(\text{Ч}))$. ∎

2.3. *The structure of secondary opeators.* When $\text{Ч} \subset J^k \P$, hence $\text{Ч}_\infty \subset J^\infty \P$, we can give a more constructive description of the algebra $\text{Диф}(F(\text{Ч}))$. For this denote by $\text{Э}_k(\text{Ч})$, $1 \leqslant k < \infty$ the set of all operators $\Delta \in \text{Diff}_k(F(\text{Ч}))$ satisfying the conditions

1) $\Delta(1) = 0$;

2) Δ is a vertical operator

3) $[\Delta, \hat{X} \mid \text{Ч}_\infty] = 0$ for any $X \in D(M)$.

If $\text{Ч}_\infty = J^\infty \P$, then we write $\text{Э}_k(\P)$ instead of $\text{Э}_k(\text{Ч})$. Clearly $\text{Э}_k(\text{Ч})$ is an $\mathbb{R}$-linear space.

THEOREM 2.3.1. 1) $Э_k(Ч) \subset FC_k(C(Ч))$;
2) $FC_k(C(Ч)) = FC'_k(C(Ч)) \oplus FC_0(C(Ч)) \oplus Э_k(Ч)$.

Proof. 1) Since operators of the form $\hat{X}\,|\,Ч_\infty$ generate $CD(F(Ч))$, see n. 1.4, then the space $FC'(C(Ч))$ is additively generated by operators of the form $\nabla \circ \hat{X}\,|\,Ч_\infty$, where $\nabla \in \mathrm{Diff}\,(F(Ч))$. If $\Delta \in Э_k(Ч)$, then $\nabla \circ \hat{X}\,|\,Ч_\infty \circ \Delta = \nabla \circ \Delta \circ \circ \hat{X}\,|\,Ч_\infty \in FC'(C(Ч))$. Hence $FC'(C(Ч)) \circ \Delta \subset FC'(C(Ч))$ and therefore $\Delta \in \in FC_k(C(Ч))$.

2) Corollary 2 and Lemma 1.5.1 obviously implies $FC'(C(Ч)) \cap Э_k(Ч) = 0$. Represent $\Delta \in FC(C(Ч))$ in the form $\Delta = \Delta(1) + (\Delta - \Delta(1))$. As in the proof of Proposition 2.2.2 one can show that $\Delta(1) \in FC_0(C(Ч))$. Let $\Delta' = \Delta - \Delta(1)$. According to Corollary 1 of Lemma 1.5.1 $\Delta' \in FC(C(Ч))$ is presentable in the form $\Delta' = \Box + \sum_i \Box_i \circ \nabla_i$, where operators $\Box$, $\Box_i$ are vertical and $\nabla_i \in \in C\,\mathrm{Diff}\,(F(Ч))$. Since $\sum_i \Box_i \circ \nabla_i \in FC(C(Ч))$, then $\Box = \Delta' - \sum_i \Box_i \circ \nabla_i \in \in FC(C(Ч))$, hence $[\hat{X}\,|\,Ч_\infty, \Box] \in FC'(C(Ч))$ for any $X \in D(M)$. But for each function $f \in C^\infty(M) : [[\hat{X}\,|\,Ч_\infty, \Box], f] = [[\hat{X}\,|\,Ч_\infty, f], \Box] + [[f, \Box], \hat{X}\,|\,Ч_\infty] = = [[\hat{X}\,|\,Ч_\infty, f], \Box] = [X(f), \Box] = 0$. Thus, $[\hat{X}\,|\,Ч_\infty, \Box]$ belongs to $FC'(C(Ч))$ and is vertical. Therefore, $[\hat{X}\,|\,Ч_\infty, \Box] = 0$ for all $X \in D(M)$ by Corollary 2 of Lemma 1.5.1, and $\Box \in Э_k(Ч)$. ∎

COROLLARY. *If* $Ч \subset J^r(¶)$, *then*

$$\text{Диф}_k(F(Ч)) = FC_0(C(Ч)) \oplus Э_k(Ч).$$

REMARK. Due to the corollary the operators from $FC_0(C(Ч)) \oplus Э_k(Ч)$ will be called intrinsic secondary operators of the equation $Ч$.

In conclusion of the n. 2.3 we obtain relations for the coefficients of the vertical operator $\Delta \in \mathrm{Diff}\,(F(Ч))$, $\Delta(1) = 0$, which are equivalent to the condition 3) of definition of $Э_k(Ч)$.

Let a coordinate chart W with the canonical coordinates q_j, p^i_σ in N^∞_m be such that coordinates q_j and some of coordinates p^i_σ form a coordinate system on $W \cap Ч_\infty$, as in n. 1.5. Then the restriction $\bar{p}^i_\sigma$ of each coordinate function p^i_σ onto $W \cap Ч_\infty$ is a function in $\bar{q}_j$, $\bar{p}^i_{\bar{\sigma}}$.

Since the operator Δ is vertical, it is presentable in the form

$$\Delta = \sum_{s=1}^{k} \sum_{\substack{i_1,\dots,i_s \\ \bar{\sigma}_1,\dots,\bar{\sigma}_s}} a^{i_1\dots i_s}_{\bar{\sigma}_1\dots\bar{\sigma}_s} \frac{\partial^s}{\partial \bar{p}^{i_1}_{\bar{\sigma}_1} \dots \partial \bar{p}^{i_s}_{\bar{\sigma}_s}}$$

in coordinates $\bar{q}_j$, $\bar{p}^i_{\bar{\sigma}}$ (see 1.5). Without loss of generality assume that coefficients

$a^{i_1\dots i_s}_{\bar\sigma_1\dots\bar\sigma_s}$ are symmetric, i.e. $a^{i_{g(1)}\dots i_{g(s)}}_{\bar\sigma_{g(1)}\dots\bar\sigma_{g(s)}} = a^{i_1\dots i_s}_{\bar\sigma_1\dots\bar\sigma_s}$ for any permutation g of s elements.

The condition $[\Delta, \hat{X} \mid Ч_\infty] = 0$ for any $X \in D(M)$ is equivalent to the condition $[\bar{D}_j, \Delta] = 0$ for any $j = 1, 2, \dots, n$ in coordinates $\bar{q}_j$, $\bar{p}^i_{\bar\sigma}$. Indeed, since in coordinates q_j a vector field $X \in D(M)$ is presentable in the form $X = \sum_{i=1}^n f \cdot \frac{\partial}{\partial q_i}$, where $f_i \in C^\infty(M)$, then $\hat{X} = \sum_{i=1}^n f_i \cdot D_i$ and $\hat{X} \mid Ч_\infty = \sum_{i=1}^n f_i \cdot \bar{D}_i$. Hence $[\hat{X} \mid Ч_\infty, \Delta] = \sum_{i=1}^n [f_i, \Delta]\, \bar{D}_i + \sum_{i=1}^n f_i [\bar{D}_i, \Delta] = \sum_{i=1}^n f_i [\bar{D}_i, \Delta]$.

Rewrite the conditions $[\bar{D}_j, \Delta] = 0$ for any $j = 1, 2, \dots, n$ it terms of coefficients $a^{i_1\dots i_s}_{\bar\sigma_1\dots\bar\sigma_s}$ of operator Δ:

$$[\bar{D}_j, \Delta] = \left[\bar{D}_j, \sum_{s=1}^k a^{i_1\dots i_s}_{\bar\sigma_1\dots\bar\sigma_s} \frac{\partial^s}{\partial \bar{p}^{i_1}_{\bar\sigma_1} \dots \partial \bar{p}^{i_s}_{\bar\sigma_s}}\right] =$$

$$= \sum_{s=1}^k \bar{D}_j(a^{i_1\dots i_s}_{\bar\sigma_1\dots\bar\sigma_s}) \cdot \frac{\partial^s}{\partial \bar{p}^{i_1}_{\bar\sigma_1} \dots \partial \bar{p}^{i_s}_{\bar\sigma_s}} +$$

$$+ \sum_{s=1}^k a^{i_1\dots i_s}_{\bar\sigma_1\dots\bar\sigma_s} \cdot \left[\bar{D}_j, \frac{\partial^s}{\partial \bar{p}^{i_1}_{\bar\sigma_1} \dots \partial \bar{p}^{i_s}_{\bar\sigma_s}}\right].$$

Since

$$\left[\bar{D}_j, \frac{\partial^s}{\partial \bar{p}^{i_1}_{\bar\sigma_1} \dots \partial \bar{p}^{i_s}_{\bar\sigma_s}}\right] =$$

$$= -\sum_{l=1}^s \sum_{1 \le r_1 < \dots < r_l \le s} \frac{\partial^l \bar{p}^t_{\bar\tau + 1_j}}{\partial \bar{p}^{i_{r_1}}_{\bar\sigma_{r_1}} \dots \partial \bar{p}^{i_{r_l}}_{\bar\sigma_{r_l}}} \cdot$$

$$\cdot \frac{\partial^{s-l+1}}{\partial \bar{p}^{i_1}_{\bar\sigma_1} \dots \widehat{\partial \bar{p}^{i_{r_1}}_{\bar\sigma_{r_1}}} \dots \widehat{\partial \bar{p}^{i_{r_l}}_{\bar\sigma_{r_l}}} \dots \partial \bar{p}^{i_s}_{\bar\sigma_s}\, \partial \bar{p}^t_{\bar\tau}},$$

then

$$\sum_{\substack{i_1,\dots,i_s \\ \sigma_1,\dots,\sigma_s}} a^{i_1\dots i_s}_{\bar\sigma_1\dots\bar\sigma_s} \left[\bar{D}_j, \frac{\partial^s}{\partial \bar{p}^{i_1}_{\bar\sigma_1} \dots \partial \bar{p}^{i_s}_{\bar\sigma_s}}\right] =$$

$$= -\sum_{l=1}^s C^l_s \frac{\partial^l \bar{p}^t_{\bar\tau + 1_j}}{\partial \bar{p}^{j_1}_{\bar\nu_1} \dots \partial \bar{p}^{j_l}_{\bar\nu_l}} \cdot a^{j_1\dots j_l i_1 \dots i_{s-l}}_{\bar\nu_1\dots\bar\nu_l \bar\sigma_1 \dots \bar\sigma_{s-l}} \frac{\partial^{s-l+1}}{\partial \bar{p}^{i_1}_{\bar\sigma_1} \dots \partial \bar{p}^{i_{s-l}}_{\bar\sigma_{s-l}} \partial \bar{p}^t_{\bar\tau}} =$$

$$= -\sum_{l=1}^{s} C_s^l \frac{1}{s-l+1} \sum_{r=1}^{s-l+1} \frac{\partial^l \bar{p}^{i_r}_{\bar{\sigma}_r+1_j}}{\partial \bar{p}^{j_1}_{\bar{\nu}_1} \ldots \partial \bar{p}^{j_l}_{\bar{\nu}_l}} \cdot$$

$$\cdot a^{j_1 \ldots j_l i_1 \ldots \hat{i}_r \ldots i_{s-l+1}}_{\bar{\nu}_1 \ldots \bar{\nu}_l \bar{\sigma}_1 \ldots \bar{\sigma}_r \ldots \bar{\sigma}_{s-l+1}} \frac{\partial^{s-l+1}}{\partial \bar{p}^{i_1}_{\bar{\sigma}_1} \ldots \partial \bar{p}^{i_{s-l+1}}_{\bar{\sigma}_{s-l+1}}} =$$

$$= -\sum_{t=1}^{s} C_s^{s-t+1} \frac{1}{t} \sum_{r=1}^{t} \frac{\partial^{s-t+1} \bar{p}^{i_r}_{\bar{\sigma}_r+1_j}}{\partial \bar{p}^{j_1}_{\bar{\nu}_1} \ldots \partial \bar{p}^{j_{s-t+1}}_{\bar{\nu}_{s-t+1}}} \cdot$$

$$\cdot a^{j_1 \ldots j_{s-t+1} i_1 \ldots \hat{i}_r \ldots i_t}_{\bar{\nu}_1 \ldots \bar{\nu}_{s-t+1} \bar{\sigma}_1 \ldots \bar{\sigma}_r \ldots \bar{\sigma}_t} \frac{\partial^t}{\partial \bar{p}^{i_1}_{\bar{\sigma}_1} \ldots \partial \bar{p}^{i_t}_{\bar{\sigma}_t}} .$$

Thus

$$[\bar{D}_j, \Delta] =$$

$$= \sum_{t=1}^{k} \left(\bar{D}_j (a^{i_1 \ldots i_t}_{\bar{\sigma}_1 \ldots \bar{\sigma}_t}) - \sum_{r=1}^{t} \frac{1}{t} \sum_{s=t}^{k} C_s^{s-t+1} \frac{\partial^{s-t+1} \bar{p}^{i_r}_{\bar{\sigma}_r+1_j}}{\partial \bar{p}^{j_1}_{\bar{\nu}_1} \ldots \partial \bar{p}^{j_{s-t+1}}_{\bar{\nu}_{s-t+1}}} \cdot \right.$$

$$\left. \cdot a^{j_1 \ldots j_{s-t+1} i_1 \ldots \hat{i}_r \ldots i_t}_{\bar{\nu}_1 \ldots \bar{\nu}_{s-t+1} \bar{\sigma}_1 \ldots \bar{\sigma}_r \ldots \bar{\sigma}_t} \right) \frac{\partial^t}{\partial \bar{p}^{i_1}_{\bar{\sigma}_1} \ldots \partial \bar{p}^{i_t}_{\bar{\sigma}_t}} =$$

$$= \sum_{t=1}^{k} \left(\bar{D}_j (a^{i_1 \ldots i_t}_{\bar{\sigma}_1 \ldots \bar{\sigma}_t}) - \sum_{r=1}^{t} \frac{1}{t} \sum_{l=1}^{k-t+1} C^l_{l+t-1} \frac{\partial^l \bar{p}^{i_r}_{\bar{\sigma}_r+1_j}}{\partial \bar{p}^{j_1}_{\bar{\nu}_1} \ldots \partial \bar{p}^{j_l}_{\bar{\nu}_l}} \cdot \right.$$

$$\left. \cdot a^{j_1 \ldots j_l i_1 \ldots \hat{i}_r \ldots i_t}_{\bar{\nu}_1 \ldots \bar{\nu}_l \bar{\sigma}_1 \ldots \bar{\sigma}_r \ldots \bar{\sigma}_t} \right) \frac{\partial^t}{\partial \bar{p}^{i_1}_{\bar{\sigma}_1} \ldots \partial \bar{p}^{i_t}_{\bar{\sigma}_t}} .$$

Note that the coefficients of $\dfrac{\partial^t}{\partial \bar{p}^{i_1}_{\bar{\sigma}_1} \ldots \partial \bar{p}^{i_t}_{\bar{\sigma}_t}}$ in the last expression are symmetric. Therefore the condition

$$[\Delta, \hat{X} \,|\, \text{Ч}_\infty] = 0 \quad \text{for any} \quad X \in D(M)$$

in terms of the coefficients is equivalent to

$$\bar{D}_j (a^{i_1 \ldots i_t}_{\bar{\sigma}_1 \ldots \bar{\sigma}_t}) = \tag{2.3.1}$$

$$= \sum_{r=1}^{t} \frac{1}{t} \sum_{l=1}^{k-t+1} C^l_{l+t-1} \frac{\partial^l \bar{p}^{i_r}_{\bar{\sigma}_r+1_j}}{\partial \bar{p}^{j_1}_{\bar{\nu}_1} \ldots \partial \bar{p}^{j_l}_{\bar{\nu}_l}} \cdot a^{j_1 \ldots j_l i_1 \ldots \hat{i}_r \ldots i_t}_{\bar{\nu}_1 \ldots \bar{\nu}_l \bar{\sigma}_1 \ldots \bar{\sigma}_r \ldots \bar{\sigma}_t}$$

for $t = 1, 2, \ldots, k; j = 1, 2, \ldots, n$ and any $\bar{\sigma}_1, \ldots, \bar{\sigma}_t, \bar{\nu}_1, \ldots, \bar{\nu}_l$.

2.4. *Generating operators*. Now, describe the operators from $\ni_k(\P)$ in canonical local coordinates q_j, p^i_σ.

Set $\mathcal{L}_j(\nabla) = [D_j, \nabla]$, where $\nabla \in \mathrm{Diff}(F(\P))$, $\mathcal{L}_\sigma = \mathcal{L}_1^{i_1} \circ \ldots \circ \mathcal{L}_n^{i_n}$, $\sigma = (i_1, i_2, \ldots, i_n)$.

THEOREM 2.4.1. 1) *An operator of the form*

$$\Delta = \sum_{i=1}^{m} \sum_{\sigma} \mathcal{L}_\sigma(\nabla_i) \frac{\partial}{\partial p^i_\sigma}, \tag{2.4.1}$$

where ∇_i are vertical operators from $\mathrm{Diff}(F(\P))$ *for $i = 1, 2, \ldots, m$, is an operator from $\ni_k(\P)$;*

2) *Each operator $\Delta \in \ni_k(\P)$ is presentable in canonical coordinates q_j, p^i_σ in the form (2.4.1).*

Proof. To prove this we need the identity

$$\begin{aligned}\left[D_j, \sum_{\substack{i_1,\ldots,i_r\\ \sigma_1,\ldots,\sigma_r}} a^{i_1\ldots i_r}_{\sigma_1\ldots\sigma_r} \frac{\partial^r}{\partial p^{i_1}_{\sigma_1} \ldots \partial p^{i_r}_{\sigma_r}}\right] &= \\ = \sum_{\substack{i_1,\ldots,i_r\\ \sigma_1,\ldots,\sigma_r}} \left(D_j(a^{i_1\ldots i_r}_{\sigma_1\ldots\sigma_r}) - \sum_{s=1}^{r} a^{i_1\ldots i_s\ldots i_r}_{\sigma_1\ldots\sigma_s+1_j\ldots\sigma_r}\right) &\frac{\partial^r}{\partial p^{i_1}_{\sigma_1} \ldots \partial p^{i_r}_{\sigma_r}},\end{aligned} \tag{2.4.2}$$

which immediately follows from

$$\left[D_j, \frac{\partial^r}{\partial p^{i_1}_{\sigma_1} \ldots \partial p^{i_r}_{\sigma_r}}\right] = -\sum_{s=1}^{r} \frac{\partial^r}{\partial p^{i_1}_{\sigma_1} \ldots \partial p^{i_s}_{\sigma_s - 1_j} \ldots \partial p^{i_r}_{\sigma_r}}, \tag{2.4.3}$$

which in its turn follows from the obvious identity

$$\left[D_j, \frac{\partial}{\partial p^i_\sigma}\right] = -\frac{\partial}{\partial p^i_{\sigma-1_j}} = \begin{cases} \dfrac{\partial}{\partial p^i_{(l_1,\ldots,l_j-1,\ldots,l_n)}}, & \text{if } \ l_j \geqslant 1 \\ 0, & \text{if } \ l_j = 0 \end{cases}$$

1) In the proof of Theorem 2.3.1 it was shown, that if ∇ is a vertical operator, then so is $[\nabla, \hat{X}]$ for every $X \in D(M)$. Therefore $\mathcal{L}_j(\nabla)$ is a vertical. Since $\dfrac{\partial}{\partial p^i_\sigma}$ is vertical, also then so is $\sum_{i,\sigma} \mathcal{L}_\sigma(\nabla_i) \dfrac{\partial}{\partial p^i_\sigma}$.

It was shown in n. 2.3 that the condition $[\Delta, \hat{X}] = 0$ for any $X \in D(M)$ is equivalent to the condition $[\Delta, D_i] = 0$ for $i = 1, 2, \ldots, n$. Further,

$$[\Delta, D_j] = \left[\sum_{i,\sigma} \mathscr{L}_\sigma(\nabla_i) \frac{\partial}{\partial p^i_\sigma}, D_j\right] =$$

$$= \sum_{i,\sigma} \left([\mathscr{L}_\sigma(\nabla_i), D_j] \circ \frac{\partial}{\partial p^i_\sigma} + \mathscr{L}_\sigma(\nabla_i) \circ \left[\frac{\partial}{\partial p^i_\sigma}, D_j\right]\right) =$$

$$= \sum_{i,\sigma} \left(- \mathscr{L}_{\sigma+1_j}(\nabla_i) \circ \frac{\partial}{\partial p^i_\sigma} + \mathscr{L}_\sigma(\nabla_i) \circ \frac{\partial}{\partial p^i_{\sigma-1_j}}\right) =$$

$$= \sum_{i,\sigma} \left(- \mathscr{L}_{\sigma+1_j}(\nabla_i) \frac{\partial}{\partial p^i_\sigma} + \mathscr{L}_{\sigma+1_j}(\nabla_i) \frac{\partial}{\partial p^i_\sigma}\right) = 0.$$

2) Let

$$\Delta = \sum_{s=1}^{k} \sum_{\substack{i_1,\ldots,i_s \\ \sigma_1,\ldots,\sigma_s}} a^{i_1\ldots i_s}_{\sigma_1\ldots\sigma_s} \frac{\partial^s}{\partial p^{i_1}_{\sigma_1} \ldots \partial p^{i_s}_{\sigma_s}}$$

be a secondary operator of order $\leqslant k$. Without loss of generality one can assume that its coefficients $a^{i_1\ldots i_s}_{\sigma_1\ldots\sigma_s}$ are symmetric. Rewrite Δ in the following form

$$\Delta = \sum_{i,\sigma} \left(\sum_{s=1}^{k} \sum_{\substack{i_1,\ldots,i_{s-1} \\ \sigma_1,\ldots,\sigma_{s-1}}} a^{i_1\ldots i_{s-1} i}_{\sigma_1\ldots\sigma_{s-1}\sigma} \frac{\partial^{s-1}}{\partial p^{i_1}_{\sigma_1} \ldots \partial p^{i_{s-1}}_{\sigma_{s-1}}}\right) \frac{\partial}{\partial p^i_\sigma} = \sum_{i,\sigma} \Delta_{i,\sigma} \frac{\partial}{\partial p^i_\sigma} .$$

The symmetry of coefficients of Δ and (2.4.1) imply that the condition $[D_j, \Delta] = 0$ for any $j = 1, 2, \ldots, n$ on Δ is equivalent to the fact that coefficients of Δ satisfy

$$\mathscr{D}_j(a^{i_1\ldots i_s}_{\sigma_1\ldots\sigma_s}) - \sum_{r=1}^{s} a^{i_1\ldots i_r\ldots i_s}_{\sigma_1\ldots\sigma_r+1_j\ldots\sigma_s} = 0 \tag{2.4.4}$$

for $s = 1, 2, \ldots, k; j = 1, 2, \ldots, n; i_1, \ldots, i_s = 1, 2, \ldots, m$, and any $\sigma_1, \ldots, \sigma_s$.

Rewrite this equation in the following form

$$a^{i_1\ldots i_{s-1} i_s}_{\sigma_1\ldots\sigma_{s-1}\sigma_s+1_j} = D_j(a^{i_1\ldots i_s}_{\sigma_1\ldots\sigma_s}) - \sum_{r=1}^{s-1} a^{i_1\ldots i_r\ldots i_s}_{\sigma_1\ldots\sigma_r+1_j\ldots\sigma_s} \tag{2.4.5}$$

for $s = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n$; $i_1, \ldots, i_s = 1, 2, \ldots, m$, and any $\sigma_1, \ldots, \sigma_s$.

Then (2.4.1) implies $\Delta_{i,\sigma+1_j} = [D_j, \Delta_{i,\sigma}]$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Hence

$$\Delta = \sum_{i,\sigma} \mathscr{L}_\sigma(\Delta_{i,0}) \frac{\partial}{\partial p^i_\sigma}. \qquad \blacksquare$$

DEFINITION. A secondary operator

$$\sum_{i,\sigma} \mathscr{L}_\sigma(\nabla_i) \frac{\partial}{\partial p^i_\sigma}$$

is denoted by $\ni_\nabla$ and the set $\nabla = (\nabla_1, \ldots, \nabla_m)$ is called the generating operator for $\ni_\nabla$.

REMARK 1. If $\nabla_i = \mathscr{S}_i \in F(\P)$, then

$$\mathscr{L}_\sigma(\mathscr{S}_i) = D_\sigma(\mathscr{S}_i) = D_1^{i_1}(\ldots(D_n^{i_n}(\mathscr{S}_i)\ldots), \quad i = 1, 2, \ldots, m$$

and $\ni_\nabla = \sum_{i,\sigma} D_\sigma(\mathscr{S}_i) \dfrac{\partial}{\partial p^i_\sigma}$ is nothing but the standard expression of evolution differentiation $\ni_{\mathscr{S}}$ in coordinates q_j, p^i_σ (see [5]), where $\mathscr{S} = (\mathscr{S}_1, \ldots, \mathscr{S}_m) = \nabla$.

REMARK 2. Generating operator for a secondary operator is not uniquely determined. For instance, let $\P = \mathbb{1}_{\mathbb{R}}$ be the trivial one-dimensional bundle over $\mathbb{R}$, $\nabla = \dfrac{\partial}{\partial p_1}$. Then $\ni_\nabla = 0 = \ni_0$.

REMARK 3. In the coordinates any secondary operator is defined by a set of its coefficients. Clearly, these coefficients are not uniquely determined if the operator is of order $\geqslant 2$. Only symmetric coefficients are uniquely determined. Coefficients $a^{i_1\ldots i_s}_{\sigma_1\ldots\sigma_s}$ of the secondary operator $\ni_\nabla$ can be uniquely recovered from $a^{i_1\ldots i_{s-1}i}_{\sigma_1\ldots\sigma_{s-1}}$ of the generating operator $\nabla = (\nabla_1, \ldots, \nabla_m)$, where

$$\nabla_i = \sum_{s=0}^{k-1} \sum_{\substack{i_1,\ldots,i_{s-1}\\ \sigma_1,\ldots,\sigma_{s-1}}} a^{i_1\ldots i_{s-1}i}_{\sigma_1\ldots\sigma_{s-1}} \frac{\partial}{\partial p^{i_1}_{\sigma_1}\ldots\partial p^{i_{s-1}}_{\sigma_{s-1}}}$$

by recursion (2.4.5) setting $a^{i_1\ldots i_{s-1}i}_{\sigma_1\ldots\sigma_{s-1}\sigma} = a^{i_1\ldots i_{s-1}i}_{\sigma_1\ldots\sigma_{s-1}}$. It what follows we will always assume that coefficients of operators $\ni_\nabla$ are obtained this way.

2.5. *Extrinsic secondary operators*. In the theory of infinitesimal symmetries of differential equations the following fact is known: each intrinsic infinitesimal

symmetry of an equation Ч may be extended to an extrinsic symmetry (see [3]). Since intrinsic infinitesimal symmetries are intrinsic secondary operators of an equation (see n. 2.2), it is natural to assume that each intrinsic secondary operator is extendable to a secondary operator on $N_m^\infty(J^\infty¶)$.
Here an elucidation is required. By definition each intrinsic secondary operator Δ of an equation is the coset $\Delta = \bar{\Box} + FC'(\mathcal{C}(\text{Ч}))$, $\bar{\Box} \in FC(\mathcal{C}(\text{Ч}))$. Since each operator $\bar{\nabla} \in FC'(\mathcal{C}(\text{Ч}))$ is presentable in the form $\bar{\nabla} = \sum_i \bar{\nabla}_i \circ (X_i \mid \text{Ч}_\infty)$, where $X_i \in \mathcal{C}\ \mathrm{Diff}\,(F(N))$, $\bar{\nabla}_i \in \mathrm{Diff}\,(F(\text{Ч}))$, it is extendable to the operator $\nabla = \sum_i \nabla_i \circ$ $\circ X_i \in FC'(\mathcal{C}(N))$, where ∇_i is an extension of $\bar{\nabla}_i$. That is why we will take as the extension of $\Delta \in \text{Диф}\,(F(\text{Ч}))$ the class $\Box + FC'(\mathcal{C}(N))$, where $\Box$ is the extension of a representative of the class Δ from Ч_∞ to N_m^∞. This definition implies that the above assumption on extendability of intrinsic secondary operators is equivalent to the assumption on extendability of any operator from $FC(\mathcal{C}(\text{Ч}))$ to an operator from $FC(\mathcal{C}(N))$. However, the following example shows that this assumption fails in such a generality.

Example. Let $¶ = \mathbb{1}_{\mathbb{R}}$, i.e. the trivial one-dimensional bundle over $\mathbb{R}$ and q, p, $p_1, \ldots, p_k, \ldots$ canonical coordinates on $J^\infty \mathbb{1}_{\mathbb{R}}$. Consider an elementary differential equation $\text{Ч} = \{p_1 = 0\} \subset J^1\,\mathbb{1}_{\mathbb{R}}$. Clearly $\text{Ч}_\infty = \{p_k = 0 \text{ for any } k = 1, 2, \ldots\}$. Hence Ч_∞ is a two-dimensional coordinate plane with coordinates q, p in $J^\infty \mathbb{1}_{\mathbb{R}}$. Clearly, the operator $a + b\ \dfrac{\partial}{\partial p}$, where functions a and b depend only on p, commutes with $\bar{D} = \dfrac{\hat{\partial}}{\partial q}\,|\,\text{Ч}_\infty = \dfrac{\partial}{\partial q}$. Hence (see n. 2.2), $a + b\ \dfrac{\partial}{\partial p} \in FC(\mathcal{C}(\text{Ч}))$. On the other hand, $\Delta(1) = \text{const}$ for each $\Delta \in FC(\mathcal{C}(N))$ (see n. 2.2). Therefore the operator $a + b\ \dfrac{\partial}{\partial p}$ is not extandable on $J^\infty \mathbb{1}_{\mathbb{R}}$ if $a \neq \text{const}$.

DEFINITION. A secondary operator $\Delta \in \text{Диф}\,(F(N))$ is an extrinsic secondary operator of an equation Ч, if the coset of Δ contains a representative, which admits a restriction onto Ч_∞. If Δ is an extrinsic secondary operator of Ч, set $\Delta\,|\,\text{Ч}_\infty = \Box\,|\,\text{Ч}_\infty + FC'(\mathcal{C}(\text{Ч}))$, where $\Box$ is a representative of the class Δ, admitting a restriction onto Ч_∞.

Now we are going to prove, that intrinsic secondary operators with constant free terms are extendable to extrinsic secondary operators.

To prove it we will need the following technical result.

Let W be a chart with the canonical coordinates q_j, p_σ^i, as in n. 1.5, so that coordinates $\bar{q}_j$, $\bar{p}_{\bar{\sigma}}^i$ constitute a coordinate system on $W \cap \text{Ч}_\infty$ and the prolongation Ч_∞ is defined by equations

$$p^i_{\underline{\sigma}} - f^i_{\underline{\sigma}}(q_j, p^i_{\bar{\sigma}}) = 0.$$

Then in W the ideal $I(\text{Ч}_\infty)$ is generated by the functions

$$\Phi^i_{\underline{\sigma}} = p^i_{\underline{\sigma}} - f^i_{\underline{\sigma}}(q_j, p^i_{\bar{\sigma}}).$$

Set $\quad f^i_{\bar{\sigma}}(q_j, p^i_{\bar{\sigma}}) \equiv p^i_{\bar{\sigma}} \quad$ and $\quad \Phi^i_{\bar{\sigma}} = p^i_{\bar{\sigma}} - f^i_{\bar{\sigma}}(q_j, p^i_{\bar{\sigma}}) \equiv 0.$

Then we may assume that in W the ideal $I(\text{Ч}_\infty)$ is generated by all the functions

$$\Phi^i_{\sigma} = p^i_{\sigma} - f^i_{\sigma}(q_j, p^i_{\bar{\sigma}}).$$

Consider the operator

$$\Box = \sum_{s=0}^{k} \sum_{\substack{i_1,\ldots,i_s \\ \sigma_1,\ldots,\sigma_s}} A^{i_1\ldots i_s}_{\sigma_1\ldots\sigma_s} \frac{\partial^s}{\partial p^{i_1}_{\sigma_1} \ldots \partial p^{i_s}_{\sigma_s}}.$$

Without loss of generality assume that its coefficients are symmetric. For simplicity denote by σ multiindices $\binom{i}{\sigma}$ of coordinate functions, operator coefficients, etc. . In these notations $\Box$ is of the form

$$\Box = \sum_{s=0}^{k} \sum_{\sigma_1,\ldots,\sigma_s} A_{\sigma_1\ldots\sigma_s} \frac{\partial^s}{\partial p_{\sigma_1} \ldots \partial p_{\sigma_s}}.$$

LEMMA 2.5.1. *The operator $\Box$ is restrictable onto Ч_∞ if and only if its coefficients satisfy*

$$(2.5.1) \qquad A_{\sigma_1\ldots\sigma_{r-1}\sigma_r}\big|\text{Ч}_\infty = -\frac{1}{r}\sum_{l=1}^{k-r+1} C^l_{l+r-1} \frac{\partial^l p_{\sigma_r}}{\partial p_{\bar{\nu}_1}\ldots\partial p_{\bar{\nu}_l}} \cdot A_{\bar{\nu}_1\ldots\bar{\nu}_l\sigma_1\ldots\sigma_{r-1}}\big|\text{Ч}_\infty$$

for $r = 1, 2, \ldots, k$; and any $\sigma_1, \ldots, \sigma_r$.

Proof. Recall that the restrictibility of $\Box$ onto Ч_∞ means that $\Box(I(\text{Ч}_\infty)) \subset I(\text{Ч}_\infty)$ and the latter is equivalent to condition $\Box(\mathcal{S})\,|\,\text{Ч}_\infty = 0$ for any $\mathcal{S} \in I(\text{Ч}_\infty)$. Let $\mathcal{S} \in I(\text{Ч}_\infty)$. Then on W the function $\mathcal{S}$ is presentable by the sum $\mathcal{S} = \sum_\sigma \lambda_\sigma \cdot \Phi_\sigma$. Therefore it is sufficient to determine when $\Box(\mathcal{S})\,|\,\text{Ч}_\infty = 0$ for $\mathcal{S} = \lambda\,\Phi_\sigma$, and any $\lambda \in F(N)$ and σ.

$$\Box(\mathcal{S}) = \sum_{s=0}^{k} A_{\sigma_1\ldots\sigma_s} \frac{\partial^s(\lambda\cdot\Phi_\sigma)}{\partial p_{\sigma_1}\ldots\partial p_{\sigma_s}} =$$

$$= \sum_{s=0}^{k} A_{\sigma_1 \ldots \sigma_s} \sum_{l=0}^{s} \sum_{1 \leqslant i_1 < \ldots < i_l \leqslant s} \frac{\partial^l \Phi_\sigma}{\partial p_{\sigma_{i_1}} \ldots \partial p_{\sigma_{i_l}}} \cdot$$

$$\cdot \frac{\partial^{s-l} \lambda}{\partial p_{\sigma_1} \ldots \widehat{\partial p_{\sigma_{i_1}}} \ldots \widehat{\partial p_{\sigma_{i_l}}} \ldots \partial p_{\sigma_s}} =$$

$$= \sum_{s=0}^{k} \sum_{l=0}^{s} C_s^l \frac{\partial^l \Phi_\sigma}{\partial p_{\nu_1} \ldots \partial p_{\nu_l}} \cdot A_{\nu_1 \ldots \nu_l \sigma_1 \ldots \sigma_{s-l}} \frac{\partial^{s-l} \lambda}{\partial p_{\sigma_1} \ldots \partial p_{\sigma_{s-l}}} =$$

$$= \sum_{r=0}^{k} \sum_{l=0}^{k-r} C_{r+l}^l \frac{\partial^l \Phi_\sigma}{\partial p_{\nu_1} \ldots \partial p_{\nu_l}} \cdot A_{\nu_1 \ldots \nu_l \sigma_1 \ldots \sigma_r} \frac{\partial^r \lambda}{\partial p_{\sigma_1} \ldots \partial p_{\sigma_r}} .$$

Since $\Phi_\sigma | Ч_\infty = 0$, then

$$\square(\mathcal{S}) | Ч_\infty = \sum_{r=0}^{k-1} \sum_{l=1}^{k-r} C_{r+l}^l \frac{\partial^l \Phi_\sigma}{\partial p_{\nu_1} \ldots \partial p_{\nu_l}} \cdot$$

$$\cdot A_{\nu_1 \ldots \nu_l \sigma_1 \ldots \sigma_r} \frac{\partial^r \lambda}{\partial p_{\sigma_1} \ldots \partial p_{\sigma_r}} | Ч_\infty .$$

Therefore $\square$ is restrictable onto $Ч_\infty$ if and only if

$$\sum_{l=1}^{k-r} C_{l+r}^l \frac{\partial^l \Phi_\sigma}{\partial p_{\nu_1} \ldots \partial p_{\nu_l}} A_{\nu_1 \ldots \nu_l \sigma_1 \ldots \sigma_r} | Ч_\infty = 0$$

for $r = 0, 1, 2, \ldots, k-1$; and any $\sigma_1, \ldots, \sigma_r, \sigma$.

Rewrite the last equation in the form

$$A_{\sigma \sigma_1 \ldots \sigma_r} | Ч_\infty = \frac{1}{r+1} \sum_{l=1}^{k-r} C_{l+r}^l \frac{\partial^l f_\sigma}{\partial p_{\bar{\nu}_1} \ldots \partial p_{\bar{\nu}_l}} A_{\bar{\nu}_1 \ldots \bar{\nu}_l \sigma_1 \ldots \sigma_r} | Ч_\infty$$

for $r = 0, 1, \ldots, k-1$ and any $\sigma_1, \ldots, \sigma_r, \sigma$ taking the form of functions Φ_σ into account. Denoting $r+1$ by r in this expression we get (2.5.1). ∎

Note that formulas (2.5.1) give recursive expression for the coefficients $A_{\sigma_1 \ldots \sigma_s} | Ч_\infty$ in terms of the coefficients $A_{\bar{\sigma}_1 \ldots \bar{\sigma}_r} | Ч_\infty$ for $s, r = 1, 2, \ldots, k$.

THEOREM 2.5.1. *Let* $Ч_\infty \subset N_m^\infty$. *If* $\Delta \in \text{ДиФ}_k(F(Ч))$ *and* $\Delta(1) = \text{const}$, *then* Δ *is extendable to an extrinsic secondary operator of the equation* Ч.

Proof. Let W be the same chart with coordinates q_j, p^i_σ as in Lemma 2.5.1. To prove the theorem it suffices to show, that in each chart W there exists an operator $\square_W \in FC(C(N))$, restrictable onto $Ч_\infty$ and such that

$$\square_W \,|\, Ч_\infty + FC'(C(Ч)) = \Delta.$$

Further, making use of the partition of unity we get the complete statement of the theorem.

In the chart $Ч_\infty \cap W$ the operator Δ is presentable in the form $\Delta = FC'(C(Ч)) + \overline{\square}$, where $\overline{\square} \in FC_0(C(Ч)) \oplus Э_k(Ч)$, due to Theorem 2.3.1. Thanks to the above we should extend $\overline{\square}$ to the operator $\square_W \in FC(C(N))$.

Let the operator $\overline{\square}$ in coordinates $\overline{q}_j, \overline{p}^i_{\overline{\sigma}}$ on $Ч_\infty \cap W$ be the form

$$\overline{\square} = a_0 + \sum_{s=1}^{k} \sum_{\overline{\sigma}_1, \dots, \overline{\sigma}_s} a_{\overline{\sigma}_1 \dots \overline{\sigma}_s} \frac{\partial^s}{\partial \overline{p}_{\overline{\sigma}_1} \dots \partial \overline{p}_{\overline{\sigma}_s}},$$

where $a_0 = \text{const}$ and $a_{\overline{\sigma}_1 \dots \overline{\sigma}_s}$ are symmetric. Making use of (2.5.1) we recursively recover $\overline{b}_{\sigma_1 \dots \sigma_s}$ on $W \cap Ч_\infty$ from $a_{\overline{\sigma}_1 \dots \overline{\sigma}_r}$, $r > 0$. Namely set

(2.5.2) $$b_{\overline{\sigma}_1 \dots \overline{\sigma}_r} = a_{\overline{\sigma}_1 \dots \overline{\sigma}_r} \quad \text{for} \quad r = 1, 2, \dots, k \quad \text{and any} \quad \overline{\sigma}_1, \dots, \overline{\sigma}_r$$

and

(2.5.3) $$\overline{b}_{\sigma_1 \dots \sigma_r} = \frac{1}{r} \sum_{l=1}^{k-r+1} C^l_{l+r-1} \frac{\partial^l \overline{p}_{\sigma_r}}{\partial p_{\overline{\nu}_1} \dots \partial p_{\overline{\nu}_l}} \cdot \overline{b}_{\overline{\nu}_1 \dots \overline{\nu}_l \sigma_1 \dots \sigma_{r-1}}$$

for $r = 1, 2, \dots, k$; and any $\sigma_1, \dots, \sigma_r$.

If $\sigma_1 = \overline{\sigma}_1, \dots, \sigma_r = \overline{\sigma}_r$, then (2.5.3) is clearly of the form $\overline{b}_{\overline{\sigma}_1 \dots \overline{\sigma}_r} = \overline{b}_{\overline{\sigma}_r \overline{\sigma}_1 \dots \overline{\sigma}_{r-1}}$ for $r = 1, 2, \dots, k$, and any $\overline{\sigma}_1, \dots, \overline{\sigma}_r$.

Therefore we may assume that (2.5.3) defines functions $\overline{b}_{\sigma_1 \dots \sigma_r}$ for any multiindices $\sigma_1, \dots, \sigma_r$.

Show that functions $\overline{b}_{\sigma_1 \dots \sigma_r}$ are symmetric for multiindices $\sigma_1, \dots, \sigma_r$. We prove it by induction in the number of unbarred multiindices. If there are no unbarred multiindices, then due to (2.5.2) and the symmetricity of coefficients $a_{\overline{\sigma}_1 \dots \overline{\sigma}_r}$ of $\overline{\square}$ all the functions $\overline{b}_{\overline{\sigma}_1 \dots \overline{\sigma}_r}$ are symmetric. Suppose that all functions $\overline{b}_{\sigma_1 \dots \sigma_r}$ with no more than s unbarred multiindices are symmetric. Let $\overline{b}_{\sigma_1 \dots \sigma_r}$ be a function with $s+1$ unbarred multiindices. If $\sigma_r = \overline{\sigma}_r$, then (2.5.3) implies $\overline{b}_{\sigma_1 \dots \overline{\sigma}_r} = b_{\overline{\sigma}_r \sigma_1 \dots \sigma_{r-1}}$. Thus the transition of the last barred multiindex the first place does not change $\overline{b}$. Therefore we may assume that in the last place there stands unbarred multiindex (more exactly, an underlined one) $\overline{b}_{\tau_1 \dots \underline{\tau}_r}$. Then the formula (2.5.3) for $\overline{b}_{\tau_1 \dots \underline{\tau}_r}$ and the inductive hypotheses imply that all functions

$\overline{b}_{\tau_1\ldots\underline{\tau}_r}$ with $s+1$ unbarred indices are symmetric in the first $r-1$ multiindices. Each of these functions in presentable in the form $\overline{b}_{\overline{\tau}_1\ldots\overline{\tau}_{r-(s+1)}\underline{\tau}_{r-s}\ldots\underline{\tau}_r}$. Hence, to prove the symmetricity of these functions it suffices to prove, that a) $\overline{b}_{\overline{\tau}_1\ldots\underline{\tau}_{r-1}\underline{\tau}_r} = \overline{b}_{\overline{\tau}_1\ldots\underline{\tau}_r\underline{\tau}_{r-1}}$ and b) $\overline{b}_{\overline{\tau}_1\ldots\overline{\tau}_i\ldots\overline{\tau}_{r-(s+1)}\ldots\underline{\tau}_r} = \overline{b}_{\overline{\tau}_1\ldots\underline{\tau}_r\ldots\overline{\tau}_{r-(s+1)}\ldots\overline{\tau}_i}$.

a) (2.5.3) implies

$$\overline{b}_{\tau_1\ldots\underline{\tau}_{r-1}\underline{\tau}_r} = \frac{1}{r}\sum_{l=1}^{k-r+1} C^l_{l+r-1}\frac{\partial^l \overline{p}_{\underline{\tau}_r}}{\partial\overline{p}_{\overline{\nu}_1}\ldots\partial\overline{p}_{\overline{\nu}_l}}\cdot$$

$$\cdot\,\overline{b}_{\overline{\nu}_1\ldots\overline{\nu}_l\tau_1\ldots\underline{\tau}_{r-1}} = \frac{1}{r}\sum_{l=1}^{k-r+1} C^l_{l+r-1}\frac{\partial^l \overline{p}_{\underline{\tau}_r}}{\partial\overline{p}_{\overline{\nu}_1}\ldots\partial\overline{p}_{\overline{\nu}_l}}\cdot$$

$$\cdot\left(\frac{1}{l+r-1}\sum_{t=1}^{k-l-r+2} C^t_{t+l+r-2}\frac{\partial^t \overline{p}_{\underline{\tau}_{r-1}}}{\partial\overline{p}_{\overline{\gamma}_1}\ldots\partial\overline{p}_{\overline{\gamma}_t}}\cdot\overline{b}_{\overline{\gamma}_1\ldots\overline{\gamma}_t\overline{\nu}_1\ldots\overline{\nu}_l\tau_1\ldots\tau_{r-2}}\right)$$

$$= \sum_{l=1}^{k-r+1}\sum_{t=1}^{k-l-r+2}\frac{(t+l+r-2)!}{t!\,l!\,r!}\cdot\frac{\partial^l \overline{p}_{\underline{\tau}_r}}{\partial\overline{p}_{\overline{\nu}_1}\ldots\partial\overline{p}_{\overline{\nu}_l}}\cdot$$

$$\cdot\frac{\partial^t \overline{p}_{\underline{\tau}_{r-1}}}{\partial\overline{p}_{\overline{\gamma}_1}\ldots\partial\overline{p}_{\overline{\gamma}_t}}\cdot\overline{b}_{\overline{\gamma}_1\ldots\overline{\gamma}_t\overline{\nu}_1\ldots\overline{\nu}_l\tau_1\ldots\tau_{r-2}}.$$

In exactly the same way

$$b_{\tau_1\ldots\underline{\tau}_r\underline{\tau}_{r-1}} = \sum_{t=1}^{k-r+1}\sum_{l=1}^{k-t-r+2}\frac{(t+l+r-2)!}{t!\,l!\,r!}\,\frac{\partial^t \overline{p}_{\underline{\tau}_{r-1}}}{\partial\overline{p}_{\overline{\gamma}_1}\ldots\partial\overline{p}_{\overline{\gamma}_t}}\times$$

$$\times\frac{\partial^l \overline{p}_{\underline{\tau}_r}}{\partial\overline{p}_{\overline{\nu}_1}\ldots\partial\overline{p}_{\overline{\nu}_l}}\cdot\overline{b}_{\overline{\nu}_1\ldots\overline{\nu}_l\overline{\gamma}_1\ldots\overline{\gamma}_t\tau_1\ldots\tau_{r-2}}.$$

Comparison of these expressions for $\overline{b}_{\tau_1\ldots\underline{\tau}_{r-1}\underline{\tau}_r}$ and $\overline{b}_{\tau_1\ldots\underline{\tau}_r\underline{\tau}_{r-1}}$ shows that they contain the same number of summands and the coefficients of the same terms are identical. Hence

$$\overline{b}_{\tau_1\ldots\underline{\tau}_{r-1}\underline{\tau}_r} = \overline{b}_{\tau_1\ldots\underline{\tau}_r\underline{\tau}_{r-1}}.$$

b) (2.5.3), the statement a) and the symmetricity of functions $\overline{b}$ in the first $r-1$ multiindices imply for $s>0$

$$b_{\overline{\tau}_1\ldots\underline{\tau}_r\ldots\overline{\tau}_i} = b_{\overline{\tau}_i\overline{\tau}_1\ldots\underline{\tau}_r\ldots\overline{\tau}_{r-(s+1)}\underline{\tau}_{r-s}\ldots\underline{\tau}_{r-1}} =$$

$$= b_{\overline{\tau}_1\ldots\overline{\tau}_i\ldots\overline{\tau}_{r-(s+1)}\underline{\tau}_{r-s}\ldots\underline{\tau}_r\underline{\tau}_{r-1}} = \overline{b}_{\overline{\tau}_1\ldots\overline{\tau}_i\ldots\underline{\tau}_{r-1}\underline{\tau}_r}.$$

If $s = 0$, then the possibility to transpose the last barred multiindex to the first place and the symmetricity in the first $r-1$ multiindices imply

$$\begin{aligned} &\overline{b}_{\overline{\tau}_1 \ldots \underline{\tau}_r \ldots \overline{\tau}_{r-1} \overline{\tau}_i} = \overline{b}_{\overline{\tau}_i \overline{\tau}_1 \ldots \underline{\tau}_r \ldots \overline{\tau}_{r-1}} = \ldots \\ &\ldots = \overline{b}_{\overline{\tau}_{i+1} \ldots \overline{\tau}_{r-1} \overline{\tau}_i \overline{\tau}_1 \ldots \overline{\tau}_{i-1} \underline{\tau}_r} = \overline{b}_{\overline{\tau}_1 \ldots \overline{\tau}_{r-1} \underline{\tau}_r}. \end{aligned}$$

Thus all functions $\overline{b}_{\sigma_1 \ldots \sigma_r}$ are symmetric in multiindices.

Coefficients $a_{\overline{\sigma}_1 \ldots \overline{\sigma}_r}$ of the intrinsic secondary operator $\overline{\Box}$ satisfy (2.3.1). Taking (2.5.2) and (2.5.3) into account these relations may be rewritten in the form

$$\overline{D}_j(\overline{b}_{\overline{\sigma}_1 \ldots \overline{\sigma}_r}) = \sum_{t=1}^{r} \overline{b}_{\overline{\sigma}_1 \ldots \overline{\sigma}_t + 1_j \ldots \overline{\sigma}_r} \tag{2.5.4}$$

for $r = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n$ and any $\overline{\sigma}_1, \ldots, \overline{\sigma}_r$. Now we will extend these relations to arbitrary multiindices. More exactly we will prove that

$$\overline{D}_j(\overline{b}_{\sigma_1 \ldots \sigma_r}) = \sum_{t=1}^{r} \overline{b}_{\sigma_1 \ldots \sigma_t + 1_j \ldots \sigma_r} \tag{2.5.5}$$

for $r = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n$ and any $\sigma_1, \ldots, \sigma_r$. To prove it we will make use of induction in the number of unbarred multiindices. The first step of induction is justified by the formula (2.5.4). Suppose that (2.5.5) is proved for s unbarred multiindices. Due to symmetricity we may assume that the function $\overline{b}_{\sigma_1 \ldots \sigma_r}$ with $s+1$ unbarred multiindices has an unbarred multiindex in the last place. Then (2.5.3) implies

$$\begin{aligned} \overline{D}_j(\overline{b}_{\sigma_1 \ldots \sigma_r}) = \frac{1}{r} \sum_{l=1}^{k-r+1} C_{l+r-1}^{l} \Bigg(&\overline{D}_j \left(\frac{\partial^l \overline{p}_{\sigma_r}}{\partial \overline{p}_{\overline{\nu}_1} \ldots \partial \overline{p}_{\overline{\nu}_l}} \right) \cdot \overline{b}_{\overline{\nu}_1 \ldots \overline{\nu}_l \sigma_1 \ldots \sigma_{r-1}} + \\ &+ \frac{\partial^l \overline{p}_{\sigma_r}}{\partial \overline{p}_{\overline{\nu}_1} \ldots \partial \overline{p}_{\overline{\nu}_l}} \overline{D}_j(\overline{b}_{\overline{\nu}_1 \ldots \overline{\nu}_l \sigma_1 \ldots \sigma_{r-1}}) \Bigg). \end{aligned}$$

Due to the inductive hypothesis

$$\overline{D}_j(\overline{b}_{\overline{\nu}_1 \ldots \overline{\nu}_l \sigma_1 \ldots \sigma_{r-1}}) = \sum_{t=1}^{l} \overline{b}_{\overline{\nu}_1 \ldots \overline{\nu}_t + 1_j \ldots \overline{\nu}_l \sigma_1 \ldots \sigma_{r-1}} + \sum_{t=1}^{r-1} \overline{b}_{\overline{\nu}_1 \ldots \overline{\nu}_l \sigma_1 \ldots \sigma_t + 1_j \ldots \sigma_{r-1}}.$$

Therefore

$$\frac{1}{r}\sum_{l=1}^{k-r+1} C^l_{l+r-1}\,\frac{\partial^l \bar{p}_{\sigma_r}}{\partial\bar{p}_{\bar{\nu}_1}\dots\partial\bar{p}_{\bar{\nu}_l}}\,\bar{D}_j(\bar{b}_{\bar{\nu}_1\dots\bar{\nu}_l\,\sigma_1\dots\sigma_{r-1}}) =$$

$$= \sum_{t=1}^{r-1}\bar{b}_{\sigma_1\dots\sigma_t+1_j\dots\sigma_{r-1}\,\sigma_r} + \frac{1}{r}\sum_{l=1}^{k-r+1}\sum_{t=1}^{l}\times$$

$$\times\, C^l_{l+r-1}\,\frac{\partial^l \bar{p}_{\sigma_r}}{\partial\bar{p}_{\bar{\nu}_1}\dots\partial\bar{p}_{\bar{\nu}_l}}\,\bar{b}_{\bar{\nu}_1\dots\bar{\nu}_t+1_j\dots\bar{\nu}_l\,\sigma_1\dots\sigma_{r-1}} =$$

$$= \sum_{t=1}^{r-1}\bar{b}_{\sigma_1\dots\sigma_t+1_j\dots\sigma_r} + \frac{1}{r}\sum_{l=1}^{k-r+1} l\,\times$$

$$\times\, C^l_{l+r-1}\,\frac{\partial^l \bar{p}_{\sigma_r}}{\partial\bar{p}_{\bar{\nu}_1}\dots\partial\bar{p}_{\bar{\nu}_l}}\,\bar{b}_{\bar{\nu}_1\dots\bar{\nu}_{l-1}\,\bar{\nu}_l+1_j\,\sigma_1\dots\sigma_{r-1}}.$$

Further, since

$$\bar{D}_j\circ\frac{\partial^l}{\partial\bar{p}_{\bar{\nu}_1}\dots\partial\bar{p}_{\bar{\nu}_l}} = \frac{\partial^l}{\partial\bar{p}_{\bar{\nu}_1}\dots\partial\bar{p}_{\bar{\nu}_l}}\circ\bar{D}_j + \left[\bar{D}_j,\ \frac{\partial^l}{\partial\bar{p}_{\bar{\nu}_1}\dots\partial\bar{p}_{\bar{\nu}_l}}\right] =$$

$$= \frac{\partial^l}{\partial\bar{p}_{\bar{\nu}_1}\dots\partial\bar{p}_{\bar{\nu}_l}}\circ\bar{D}_j - \sum_{t=1}^{l}\ \sum_{1\leqslant i_1<\dots<i_t\leqslant l}\frac{\partial^t \bar{p}_{\bar{\tau}+1_j}}{\partial\bar{p}_{\bar{\nu}_{i_1}}\dots\partial\bar{p}_{\bar{\nu}_{i_t}}}\times$$

$$\times\,\frac{\partial^{l-t+1}}{\partial\bar{p}_{\bar{\nu}_1}\dots\widehat{\partial\bar{p}_{\bar{\nu}_{i_1}}}\dots\widehat{\partial\bar{p}_{\bar{\nu}_{i_t}}}\dots\partial\bar{p}_{\bar{\nu}_l}\,\partial\bar{p}_{\bar{\tau}}},$$

then

$$\frac{1}{r}\sum_{l=1}^{k-r+1} C^l_{l+r-1}\bar{D}_j\left(\frac{\partial^l \bar{p}_{\sigma_r}}{\partial\bar{p}_{\bar{\nu}_1}\dots\partial\bar{p}_{\bar{\nu}_l}}\right)\cdot\bar{b}_{\bar{\nu}_1\dots\bar{\nu}_l\,\sigma_1\dots\sigma_{r-1}} =$$

$$= \frac{1}{r}\sum_{l=1}^{k-r+1} C^l_{l+r-1}\,\frac{\partial^l \bar{p}_{\sigma_r+1_j}}{\partial\bar{p}_{\bar{\nu}_1}\dots\partial\bar{p}_{\bar{\nu}_l}}\cdot\bar{b}_{\bar{\nu}_1\dots\bar{\nu}_l\,\sigma_1\dots\sigma_{r-1}} -$$

$$- \frac{1}{r}\sum_{l=1}^{k-r+1} C^l_{l+r-1}\sum_{t=1}^{l}\ \sum_{1\leqslant i_1<\dots<i_t\leqslant l}\frac{\partial^t \bar{p}_{\bar{\tau}+1_j}}{\partial\bar{p}_{\bar{\nu}_{i_1}}\dots\partial\bar{p}_{\bar{\nu}_{i_t}}}\times$$

$$\times\,\frac{\partial^{l-t+1}\bar{p}_{\sigma_r}}{\partial\bar{p}_{\bar{\nu}_1}\dots\widehat{\partial\bar{p}_{\bar{\nu}_{i_1}}}\dots\widehat{\partial\bar{p}_{\bar{\nu}_{i_t}}}\dots\partial\bar{p}_{\bar{\nu}_l}\,\partial\bar{p}_{\bar{\tau}}}\times\bar{b}_{\bar{\nu}_1\dots\bar{\nu}_l\sigma_1\dots\sigma_{r-1}} =$$

$$= \overline{b}_{\sigma_1 \ldots \sigma_{r-1} \sigma_r + 1_j} - \frac{1}{r} \sum_{l=1}^{k-r+1} \sum_{t=1}^{l} C^l_{l+r-1} C^t_l \frac{\partial^t \overline{p}_{\overline{\tau}+1_j}}{\partial \overline{p}_{\overline{\gamma}_1} \ldots \partial \overline{p}_{\overline{\gamma}_t}} \times$$

$$\times \frac{\partial^{l-t+1} \overline{p}_{\sigma_r}}{\partial \overline{p}_{\overline{\gamma}_1} \ldots \partial \overline{p}_{\overline{\gamma}_{l-t}}} \cdot \overline{b}_{\overline{\gamma}_1 \ldots \overline{\gamma}_t \overline{\nu}_1 \ldots \overline{\nu}_{l-t} \sigma_1 \ldots \sigma_{r-1}}.$$

Thus

$$\overline{D}_j(\overline{b}_{\sigma_1 \ldots \sigma_r}) = \sum_{t=1}^{r} \overline{b}_{\sigma_1 \ldots \sigma_t + 1_j \ldots \sigma_r} +$$

$$+ \left\{ \frac{1}{r} \sum_{l=1}^{k-r+1} l\, C^l_{l+r-1} \frac{\partial^l \overline{p}_{\sigma_r}}{\partial \overline{p}_{\overline{\nu}_1} \ldots \partial \overline{p}_{\overline{\nu}_l}} \cdot \overline{b}_{\overline{\nu}_1 \ldots \overline{\nu}_{l-1} \overline{\nu}_l + 1_j \sigma_1 \ldots \sigma_{r-1}} - \right.$$

$$- \frac{1}{r} \sum_{l=1}^{k-r+1} \sum_{t=1}^{l} C^l_{l+r-1} C^t_l \frac{\partial^t \overline{p}_{\overline{\tau}+1_j}}{\partial \overline{p}_{\overline{\gamma}_1} \ldots \partial \overline{p}_{\overline{\gamma}_t}} \times$$

$$\left. \times \frac{\partial^{l-t+1} \overline{p}_{\sigma_r}}{\partial \overline{p}_{\overline{\nu}_1} \ldots \partial \overline{p}_{\overline{\nu}_{l-t}} \partial \overline{p}_{\overline{\tau}}} \cdot \overline{b}_{\overline{\gamma}_1 \ldots \overline{\gamma}_t \overline{\nu}_1 \ldots \overline{\nu}_{l-t} \sigma_1 \ldots \sigma_{r-1}} \right\}.$$

Now, prove that the expression in brackets vanishes. Applying (2.5.3) to $\overline{b}_{\overline{\nu}_1 \ldots \overline{\nu}_l + 1_j \sigma_1 \ldots \sigma_{r-1}}$, we get

$$\frac{1}{r} \sum_{l=1}^{k-r+1} l C^l_{l+r-1} \frac{\partial^l \overline{p}_{\sigma_r}}{\partial \overline{p}_{\overline{\nu}_1} \ldots \partial \overline{p}_{\overline{\nu}_l}} \cdot \left(\frac{1}{l+r-1} \sum_{s=1}^{k-l-r+2} \times \right.$$

$$\left. \times C^s_{s+l+r-2} \frac{\partial^s \overline{p}_{\overline{\nu}_l+1_j}}{\partial \overline{p}_{\overline{\gamma}_1} \ldots \partial \overline{p}_{\overline{\gamma}_s}} \cdot b_{\overline{\gamma}_1 \ldots \overline{\gamma}_s \overline{\nu}_1 \ldots \overline{\nu}_{l-1} \sigma_1 \ldots \sigma_{r-1}} \right) =$$

$$= \frac{1}{r} \sum_{l=1}^{k-r+1} \sum_{s=1}^{k-l-r+2} \frac{1}{l+r-1} l C^l_{l+r-1} C^s_{s+l+r-2} \frac{\partial^l \overline{p}_{\sigma_r}}{\partial \overline{p}_{\overline{\nu}_1} \ldots \partial \overline{p}_{\overline{\nu}_l}} \times$$

$$\times \frac{\partial^s \overline{p}_{\nu_l+1_j}}{\partial \overline{p}_{\gamma_1} \ldots \partial \overline{p}_{\gamma_s}} \cdot \overline{b}_{\overline{\gamma}_1 \ldots \overline{\gamma}_s \overline{\nu}_1 \ldots \overline{\nu}_{l-1} \sigma_1 \ldots \sigma_{r-1}}.$$

Thus the expression in brackets may be rewritten as

$$\left(\frac{1}{r} \sum_{l=1}^{k-r+1} \sum_{s=1}^{k-l-r+2} \frac{l}{l+r-1} C^l_{l+r-1} C^s_{s+l+r-2} \frac{\partial^l \overline{p}_{\sigma_r}}{\partial \overline{p}_{\overline{\nu}_1} \ldots \partial \overline{p}_{\overline{\nu}_l}} \times \right.$$

$$\times \frac{\partial^s \bar{p}_{\bar{\nu}_l + 1_j}}{\partial \bar{p}_{\bar{\gamma}_1} \ldots \partial \bar{p}_{\bar{\gamma}_s}} \cdot b_{\bar{\gamma}_1 \ldots \bar{\gamma}_s \bar{\nu}_1 \ldots \bar{\nu}_{l-1} \sigma_1 \ldots \sigma_{r-1}} \Bigg) -$$

$$-\Bigg(\frac{1}{r} \sum_{l=1}^{k-r+1} \sum_{t=1}^{l} C^l_{l+r-1} C^t_l \frac{\partial^t \bar{p}_{\bar{\tau}+1_j}}{\partial \bar{p}_{\bar{\gamma}_1} \ldots \partial \bar{p}_{\bar{\gamma}_t}} \times$$

$$\times \frac{\partial^{l-t+1} \bar{p}_{\sigma_r}}{\partial \bar{p}_{\bar{\nu}_1} \ldots \partial \bar{p}_{\bar{\nu}_{l-t}} \partial \bar{p}_{\bar{\tau}}} \cdot b_{\bar{\gamma}_1 \ldots \bar{\gamma}_t \bar{\nu}_1 \ldots \bar{\nu}_{l-t} \sigma_1 \ldots \sigma_{r-1}} \Bigg).$$

Obviously, both parentheses contain the same number of summands and same summands have the same coefficients. Hence the expression in brackets vanishes. Thus (2.5.5) is true for all multiindices.

Rewrite (2.5.5) in the form

$$\bar{b}_{\sigma_1 \ldots \sigma_{r-1} \sigma_r + 1_j} = \bar{D}_j(\bar{b}_{\sigma_1 \ldots \sigma_{r-1} \sigma_r}) - \sum_{t=1}^{r-1} \bar{b}_{\sigma_1 \ldots \sigma_t + 1_j \ldots \sigma_r} \tag{2.5.6}$$

for $r = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n$; and any $\sigma_1, \sigma_2, \ldots, \sigma_r$ and observe that all functions $\bar{b}_{\sigma_1 \ldots \sigma_r}$ are recurrently expressed in terms of functions $\bar{b}_{\sigma_1 \ldots \sigma_s 0}$, where $0 = \begin{pmatrix} i \\ (0, \ldots, 0) \end{pmatrix} = \begin{pmatrix} i \\ 0 \end{pmatrix}$, $s = 1, 2, \ldots, k-1$, for any $\sigma_1, \ldots, \sigma_s$ via (2.5.6).

Now we extend each function $\bar{b}^{i_1 \ldots i_s i}_{\sigma_1 \ldots \sigma_s 0}$ in some way to a function $b^{i_1 \ldots i_s i}_{\sigma_1 \ldots \sigma_s 0}$ in the domain W and consider the operator $\nabla = (\nabla_1, \nabla_2, \ldots, \nabla_m)$, where

$$\nabla_i = \sum_{s=0}^{k} \sum_{\sigma_1, \ldots, \sigma_s} b^{i_1 \ldots i_s i}_{\sigma_1 \ldots \sigma_s 0} \frac{\partial^s}{\partial p^{i_1}_{\sigma_1} \ldots \partial p^{i_s}_{\sigma_s}}.$$

The coefficients of the secondary operator

$$\square_W = a_0 + \ni_\nabla = a_0 + \sum_{r=1}^{k} \sum_{\sigma_1, \ldots, \sigma_r} A_{\sigma_1 \ldots \sigma_r} \frac{\partial^r}{\partial p_{\sigma_1} \ldots \partial p_{\sigma_r}}$$

satisfy the relations (2.4.5):

$$A_{\sigma_1 \ldots \sigma_{r-1} \sigma_r + 1_j} = D_j(A_{\sigma_1 \ldots \sigma_r}) - \sum_{t=1}^{r-1} A_{\sigma_1 \ldots \sigma_t + 1_j \ldots \sigma_r}$$

for $r = 1, 2, \ldots, k; j = 1, 2, \ldots, n$; and any $\sigma_1, \ldots, \sigma_r$. Therefore

$$A_{\sigma_1\ldots\sigma_{r-1}\sigma_r+1_j}\,|\,\text{Ч}_\infty = \overline{D}_j(A_{\sigma_1\ldots\sigma_r}\,|\,\text{Ч}_\infty) - \sum_{t=1}^{r-1} A_{\sigma_1\ldots\sigma_t+1_j\ldots\sigma_r}\,|\,\text{Ч}_\infty$$

for $r = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n$; and any $\sigma_1, \ldots, \sigma_r$. Comparing the last relations with (2.5.6) and taking into account that

$$A_{\sigma_1\ldots\sigma_{r-1}0}\,|\,\text{Ч}_\infty = b_{\sigma_1\ldots\sigma_{r-1}0}\,|\,\text{Ч}_\infty = \overline{b}_{\sigma_1\ldots\sigma_{r-1}0}$$

we see that

(2.5.7) $\quad A_{\sigma_1\ldots\sigma_r}\,|\,\text{Ч}_\infty = \overline{b}_{\sigma_1\ldots\sigma_r}$ for $r = 1, 2, \ldots, k$; and any $\sigma_1, \ldots, \sigma_r$.

Since $\overline{b}_{\sigma_1\ldots\sigma_r}$ are symmetric, then

$$\frac{1}{r!}\,A_{(\sigma_1\ldots\sigma_r)}\,|\,\text{Ч}_\infty = \overline{b}_{\sigma_1\ldots\sigma_r}$$

for $r = 1, 2, \ldots, k$; and any $\sigma_1, \ldots, \sigma_r$ where $A_{(\sigma_1\ldots\sigma_r)} = \sum_{g\in S_r} A_{\sigma_{g(1)}\ldots\sigma_{g(r)}}$ and S_r is the permutation group of r elements. Therefore (2.5.3) is identical to conditions (2.5.1) of restrictability of $\square_W$ with symmetrized coefficients onto Ч_∞. Thus $\square_W$ is restrictable onto $\text{Ч}_\infty \cap W$. In particular, (2.5.7), implies

$$A_{(\overline{\sigma}_1\ldots\overline{\sigma}_r)}\,|\,\text{Ч}_\infty = \overline{b}_{\overline{\sigma}_1\ldots\overline{\sigma}_r} = a_{\overline{\sigma}_1\ldots\overline{\sigma}_r}$$

for $r = 1, 2, \ldots, k$, and any $\overline{\sigma}_1, \ldots, \overline{\sigma}_r$, i.e. $\overline{\square}_W\,|\,\text{Ч}_\infty \cap W = \square$. ∎

COROLLARY. *Let* $\text{Ч} \subset J^k\P$. Then any $\overline{\Delta} \in \text{Э}_k(\text{Ч})$ *is extendable to an operator* $\Delta \in \text{Э}_k(\P)$.

§3. SECONDARY DIFFERENTIAL OPERATORS. FUNCTIONAL APPROACH

3.1. Disadvantages of geometric approach to the secondary operators theory are that it is not clear on what kind of objects secondary operators act. The answer to this question seems actual especially since we have defined secondary operators as cosets of differential operators on Ч_∞ and a priori it is not clear on what objects such cosets may act. More exactly, such a coset correctly determines an $\mathbb{R}$-linear mapping of the space of functions constant on leaves of the foliation $C(\text{Ч})$ into itself. However, as a rule, this space consists of constants (see [5]) and all its $\mathbb{R}$-linear maps are operators from $\text{Диф}_0(F(\text{Ч})) = \mathbb{R}$. Thus, if we accept a too formal approach, the existence of secondary operators of non-zero order forces us to doubt whether the approach put forward in the previous section is well justified.

The simplest way of reacting in the described situation (keeping in mind virtual particles of contemporary quantum field theory and virtuality of the bundle $Ч_\infty \to \mathrm{Sol}\, Ч$) is to consider secondary operators as virtual ones, i.e. operators able to act on something only under certain favourable conditions.

However we migth attempt to make the secondary operators act not on functions but on some other objects. In this section we find such an action under another «functional» approach to constructing secondary differential operators. The idea is that from the very beginning we should determine smooth functions on the «manifold» Sol Ч and only afterwards the secondary differential operators as localizable operators acting on these functions.

3.2. A «smooth function» on the «manifold» Sol Ч is a cohomology class $\Omega \in \in \bar{H}^n(Ч_\infty)$ (see n. 1.6), where n is a number of independent variables. If L is an n-dimensional integral manifold of the Cartan distribution on $Ч_\infty$, i.e. L is a point of Sol Ч, then one can understand image the value of the «function» Ω at a «point» L as $\Omega \mid L = \int_L \omega \mid L$, where $\omega \in \bar{\Lambda}^n(Ч)$ is a horizontal form on $Ч_\infty$ representing Ω. Recall (see, e.g. [5]) that $\Omega \mid L$ is naturally considered as an element de Rham cohomology group $H^n(L)$ and Ω as the «action», i.e. an expression of the form $\int \mathscr{L}(q_j, p^i, p^i_\sigma)\, dq,\ \ dp = dq_1 \wedge \ldots \wedge dq_n$. The above point of view is motivated in many ways (see e.g. [1]). Here we draw the reader's attention to the fact that functions introduced on Sol Ч are also of virtual character, because the integration $\int_L \omega \mid L$ is in general impracticable.

Now we are to define differential operators as some maps from $\bar{H}^n(Ч)$ into itself. Here we encounter an obstacle: the group $\bar{H}^n(Ч)$ is not, in general, an $F(Ч)$-module. Therefore we can not make use of the standard algebraic definitions (see [3]). Since $\bar{H}^n(Ч) = \bar{\Lambda}^n(Ч)/\bar{d}\bar{\Lambda}^{n-1}(Ч)$ and $\bar{\Lambda}^n(Ч)$ is an $F(Ч)$-module, we can understand under a differential operator acting in $\bar{H}^n(Ч)$ a differential operator $\Delta : \bar{\Lambda}^n(Ч) \longrightarrow \bar{\Lambda}^n(Ч)$, such that

$$\Delta\,(\bar{d}\bar{\Lambda}^{n-1}(Ч)) \subset \bar{d}(\bar{\Lambda}^{n-1}(Ч)).$$

Indeed, this operator Δ naturally gives rise to a map $\bar{H}^n(Ч) \longrightarrow \bar{H}^n(Ч)$.

The above should be clarified. Namely, set

$$\overline{\mathrm{Diff}}\,(\bar{\Lambda}^n(Ч), \bar{\Lambda}^n(Ч)) = \{\Delta \in \mathrm{Diff}\,(\bar{\Lambda}^n(Ч),\ \bar{\Lambda}^n(Ч)) \mid \Delta\,(\bar{d}\bar{\Lambda}^{n-1}(Ч)) \subset \bar{d}\bar{\Lambda}^{n-1}(Ч)\},$$

$$\underline{\mathrm{Diff}}\,(\bar{\Lambda}^n(Ч), \bar{\Lambda}^n(Ч)) = \{\Delta \in \mathrm{Diff}\,(\bar{\Lambda}^n(Ч),\ \bar{\Lambda}^n(Ч)) \mid \Delta\,(\bar{\Lambda}^n(Ч)) \subset \bar{d}\bar{\Lambda}^{n-1}(Ч)\}.$$

Clearly, $\underline{\mathrm{Diff}}\,(\bar{\Lambda}^n(Ч), \bar{\Lambda}^n(Ч))$ is a two-sided ideal in the $\mathbb{R}$-algebra $\overline{\mathrm{Diff}}\,(\bar{\Lambda}^n(Ч), \bar{\Lambda}^n(Ч))$. Therefore the quotient algebra

$$\text{ДиФ}\,(\bar{H}^n(\text{Ч}), \bar{H}^n(\text{Ч})) = \overline{\mathrm{Diff}}\,(\bar{\Lambda}^n(\text{Ч}), \bar{\Lambda}^n(\text{Ч}))/\underline{\mathrm{Diff}}\,(\bar{\Lambda}^n(\text{Ч}), \bar{\Lambda}^n(\text{Ч}))$$

is defined.

DEFINITION. Elements of the algebra $\text{ДиФ}\,(\bar{H}^n(\text{Ч}), \bar{H}^n(\text{Ч}))$ are called secondary functional differential operators of the equation Ч.

3.3. Now our aim is to prove that both definitions of secondary differential operators coincide when $\text{Ч}_\infty = J^\infty ¶$. For this we will need the following results.

Each vertical operator $\Delta \in \mathrm{Diff}_k(F(¶))$ defines an operator $\tilde{\Delta} \in \mathrm{Diff}\,(\bar{\Lambda}^n(¶), \bar{\Lambda}^n(¶))$ by the formula

$$\tilde{\Delta}(f \cdot \bar{\omega}_0) = \Delta(f) \cdot \bar{\omega}_0,$$

where $f \in F(¶)$, $\bar{\omega}_0 \in \bar{\Lambda}^n(¶)$ and $\bar{\omega}_0$ is a local volume form on the manifold M. Since Δ is vertical, $\tilde{\Delta}$ is clearly well defined.

LEMMA 3.3.1. *Let* $\Delta \in \mathrm{Diff}\,(F(¶))$ *be a vertical operator. If* $\mathrm{im}\,\tilde{\Delta} \subset \bar{d}\bar{\Lambda}^{n-1}(¶)$, *then* $\Delta = 0$.

Proof. Let $\mathrm{im}\,\Delta \subset \bar{d}\bar{\Lambda}^{n-1}(¶)$. Then $\mathscr{E}\,(\Delta(\bar{\omega})) = 0$ for each form $\bar{\omega} \in \bar{\Lambda}^n(¶)$, where $\mathscr{E}$ is the Euler operator. Let $f \in F(¶)$ and $g \in C^\infty(M) \subset F(¶)$. Then

$$\tilde{\Delta}(g \cdot f \cdot \bar{\omega}_0) = \Delta(g \cdot f)\,\bar{\omega}_0 = g\,\Delta(f) \cdot \bar{\omega}_0$$

and

$$0 = \mathscr{E}\,(\tilde{\Delta}(g \cdot f \cdot \bar{\omega}_0)) = l^*_{\tilde{\Delta}(g \cdot f \cdot \bar{\omega}_0)}(1) = l^*_{g \cdot \Delta(f) \cdot \bar{\omega}_0}(1) = l^*_{\Delta(f)\bar{\omega}_0}(g).$$

Since operators $l^*_{\bar{\omega}}$ are C-differential, $l^*_{\Delta(f) \cdot \bar{\omega}_0} \equiv 0$. Therefore $l_{\Delta(f)\bar{\omega}_0} \equiv 0$, too. The latter is equivalent to $\Delta(f) \in C^\infty(M)$. Thus we get $\Delta(f) \in C^\infty(M)$ for any $f \in F(¶)$. Then $\dfrac{\partial}{\partial p^i_\sigma}(\Delta(f)) = 0$ for each coordinate function p^i_σ, i.e. $\dfrac{\partial}{\partial p^i_\sigma} \circ \Delta \equiv 0$. It clearly implies $\Delta = 0$. ∎

Each operator $\Delta \in Э_k(¶)$ is vertical, therefore it defines the operator $\tilde{\Delta} \in \mathrm{Diff}_k(\bar{\Lambda}^n(¶), \bar{\Lambda}^n(¶))$. Denote the space of these operators by $\tilde{Э}_k(¶)$.

THEOREM 3.3.1. 1) *The space* $\tilde{Э}_k(¶)$ *is a subspace of* $\overline{\mathrm{Diff}}_k(\bar{\Lambda}^n(¶), \bar{\Lambda}^n(¶))$, $1 \geqslant$ $\geqslant k < \infty$;

2) $\overline{\mathrm{Diff}}_k(\bar{\Lambda}^n(¶), \bar{\Lambda}^n(¶)) = \underline{\mathrm{Diff}}_k(\bar{\Lambda}^n(¶), \bar{\Lambda}^n(¶)) \oplus \mathbb{R} \oplus \tilde{Э}_k(¶)$.

Proof. It suffices to prove both statements in a chart with canonical coordinates

q_j, p^i_σ. In these coordinates each element from $\bar{\mathrm{d}}\bar{\Lambda}^{n-1}(\P)$ is presentable in the form $\sum_{j=1}^{n} D_j(\mathscr{S}_j)\,\bar{\omega}_0$, where $\mathscr{S}_j \in F(\P)$, $\bar{\omega}_0 = \mathrm{d}q_1 \wedge \ldots \wedge \mathrm{d}q_n$. Since any operator $\Delta \in \ni_k(\P)$ commutes with D_j, $j = 1, 2, \ldots, n$, then

$$\tilde{\Delta}\left(\sum_{j=1}^{n} D_j(\mathscr{S}_j)\bar{\omega}_0\right) = \sum_{j=1}^{n} \Delta(D_j(\mathscr{S}_j))\bar{\omega}_0 = \sum_{j=1}^{n} D_j(\Delta(\mathscr{S}_j))\bar{\omega}_0 .$$

The first statement is proved.

To prove the second statement write the operator $\Delta \in \overline{\mathrm{Diff}}\,(\bar{\Lambda}^n(\P), \bar{\Lambda}^n(\P))$ in coordinates q_j, p^i_σ:

$$\Delta = \bar{\omega}_0 \otimes \sum_{s+|\tau|=0}^{k} a^{i_1\ldots i_s}_{\sigma_1\ldots\sigma_s\tau} \frac{\partial^{s+|\tau|}}{\partial p^{i_1}_{\sigma_1}\ldots\partial p^{i_s}_{\sigma_s}\,\partial q_\tau} = \bar{\omega}_0 \otimes \Delta_{\bar{\omega}_0} .$$

Due to Remark of n. 1.5 $\Delta_{\bar{\omega}_0}$ is uniquely presentable in the form

$$\Delta_{\bar{\omega}_0} = \sum_{|\sigma| \geqslant 0} D_\sigma \circ \Box_\sigma$$

where $\Box_\sigma$ are vertical operators. Set

$$\nabla_{\bar{\omega}_0} = \sum_{|\sigma| > 0} D_\sigma \circ \Box_\sigma .$$

Then $\Delta = \nabla + \Box$. We have $\nabla \in \underline{\mathrm{Diff}}\,(\bar{\Lambda}^n(\P), \bar{\Lambda}^n(\P))$. Indeed, the condition $\nabla'(\bar{\Lambda}^n) \subset \subset \bar{\mathrm{d}}\bar{\Lambda}^{n-1}(\P)$ is equivalent to the condition $\nabla'_{\bar{\omega}_0}(F(\P)) \subset \sum_{j=1}^{n} D_j(F(\P))$ verified by $\nabla_{\bar{\omega}_0}$. Since Δ, $\nabla \in \overline{\mathrm{Diff}}\,(\bar{\Lambda}^n(\P), \bar{\Lambda}^n(\P))$, then $\Box = \Delta - \nabla \in \overline{\mathrm{Diff}}\,(\bar{\Lambda}^n(\P), \bar{\Lambda}^n(\P))$ or equivalenty $\Box_{\bar{\omega}_0}\left(\sum_{j=1}^{n} D_j(F(\pi))\right) \subset \sum_{j=1}^{n} D_j(F(\pi))$.

But the latter is equivalent to $[D_j, \Box_{\bar{\omega}_0}]\,(F(\P)) \subset \sum_{j=1}^{n} D_j(F(\P))$ for any $j = 1, 2, \ldots, n$. Now Lemma 3.3.1 implies $[D_j, \Box_{\bar{\omega}_0}] = 0$ for any $j = 1, 2, \ldots, n$, since $[D_j, \Box_{\bar{\omega}_0}]$ is a vertical operator. Therefore $\Box_{\bar{\omega}_0} \in \mathbb{R} \oplus \ni_k(\P)$ and $\Box \in \mathbb{R} \oplus \oplus \tilde{\ni}_k(\P)$. ■

COROLLARY. $\text{ДиФ}_k(\bar{H}^n(\P), \bar{H}^n(\P)) = \tilde{\ni}_k(\P) \oplus \mathbb{R}$.

REMARK. A trivial example of euqation $\text{Ч} = \{p_1 = 0\} \subset \mathscr{T}^1\,\mathbb{1}_{\mathbb{R}}$ shows that if $\text{Ч}_\infty \neq \mathscr{T}^\infty\P$, then $\text{ДиФ}_k(\bar{H}^n(\text{Ч}), \bar{H}^n(\text{Ч})) \neq \tilde{\ni}_k(\text{Ч}) \oplus \mathbb{R}$ in general.

3.4. One may define secondary differential operators of infinite order literally

following the above geometrical or functional definition of the finite order secondary operators. We mean that an infinite order differential operator is an $\mathbb{R}$-linear map $\Delta : F(\text{Ч}) \longrightarrow F(\text{Ч})$ satisfying the following conditions:
1) $\Delta(F_i(\text{Ч})) \subset F_{i+j}(\text{Ч})$, $j = j(i)$; 2) $\Delta \,|\, F_i(\text{Ч})$ is a differential operator of order $k = k(i)$ and $k(0) \leqslant k(1) \leqslant k(2) \leqslant \ldots$. It is quite clear that all results of this paper are true in the case of the infinite order secondary operators.

§4. APPROXIMATION OF SECONDARY OPERATORS BY AVOLUTION DIFFERENTIATIONS

4.1. It is well known that each scalar differential k-th order operator on a smooth manifold is presentable as a sum of compositions of 1^{st} order operators and a free term. Does this fact hold for secondary operators? The following example shows, that the answer to this question is negative.

Example. Let $\P = \mathbb{1}_{\mathbb{R}}$, $\nabla = \sum_{i=0}^{\infty} p_{i^2} \dfrac{\partial}{\partial p_i}$, where q, p_i are canonical coordinates on $\mathcal{T}^{\infty}\mathbb{1}_{\mathbb{R}}$. Then $\ni_\nabla$ is not be presentable in the form

$$\ni_\nabla = \sum_{s=1}^{l} \ni_{\varphi_s} \circ \ni_{\psi_s} + \sum_{r=1}^{l} \ni_{f_r}. \tag{4.1.1}$$

To make sure of this, note that the coefficients of secondary operators of the form

$$\ni_\square = \sum_{i,j} a_{ij} \frac{\partial^2}{\partial p_i \, \partial p_j}$$

satisfy

$$a_{ij} = \sum_{s=0}^{j} (-1)^s C_j^s D^{j-s}(a_{i+s\,0}),$$

which follows from the recursion (2.4.5) and Remark 3 of n. 2.4. For $\ni_\nabla$ it implies

$$a_{i0} + a_{0i} = p_{i^2} + \sum_{s=0}^{i} (-1)^s \, C_i^s D^{i-s}(p_{s^2}).$$

This formula shows that functions $a_{i0} + a_{0i}$ are of the form

$$a_{i0} + a_{0i} = p_{i^2} + f(p, \ldots, p_{i^2-1}). \tag{4.1.2}$$

On the other hand, (4.1.1) implies

$$a_{i0} + a_{0i} = \sum_{s=1}^{l} (D^i(\mathscr{S}_s) \cdot \psi_s + \mathscr{S}_s D^i(\psi_s)). \tag{4.1.3}$$

Let k be the maximum number of variables p_j on which functions $\mathscr{S}_s$ and ψ_s, $s = 1, 2, \ldots, l$, depend. Then functions $a_{i0} + a_{0i}$ do not depend on variables $p_j, j > k + i$ as follows from (4.1.3). It contradicts (4.1.2).

4.2. However it turns out that any secondary operator may be locally approximated with any accuracy by a sum of compositions of secondary 1st operators. More exactly the following theorem holds.

THEOREM 4.2.1. *Let* $\Delta \in \ni_k(\P)$. *Then there exists a finite set of evolution differentiations* $\ni_{\mathscr{S}_{1,1}}, \ldots, \ni_{\mathscr{S}_{k(1),1}}; \ldots; \ni_{\mathscr{S}_{1,l}}, \ldots, \ni_{\mathscr{S}_{k(l),l}}$ *for any canonical chart* W *any positive integer* r *such that the restriction* $\Delta | F_r(\P)$ *of* Δ *is presentable on* W *in the form*

$$\Delta | F_r(\pi) = \sum_{s=1}^{l} \ni_{\mathscr{S}_{1,s}} \circ \ldots \circ \ni_{\mathscr{S}_{k(s),s}} | F_r(\pi).$$

Proof. First we will prove for any homogeneous secondary operator

$$\Delta = \sum_{\substack{i_1,\ldots,i_k \\ \sigma_1,\ldots,\sigma_k}} a^{i_1\ldots i_k}_{\sigma_1\ldots\sigma_k} \frac{\partial}{\partial p^{i_1}_{\sigma_1} \ldots \partial p^{i_k}_{\sigma_k}}$$

on W and any positive integer r there is a finite set of evolution differentiations $\ni_{\mathscr{S}_1}, \ldots, \ni_{\mathscr{S}_L}$ and a finite set of homogeneous $(k-1)$-th order secondary operators $\Delta_1, \ldots, \Delta_L$ such that

$$\Delta | F_r(\P) = \operatorname{smbl}\left(\sum_{l=1}^{L} \ni_{\mathscr{S}_l} \circ \Delta_l\right) | F_r(\P),$$

where smbl $\square$ denotes the sum of all k-th order terms of $\square$.

Let $\Delta = \ni_\nabla$, where $\nabla = (\nabla_1, \ldots, \nabla_m)$.

$$\nabla_i = \sum_{\substack{i_1,\ldots,i_{k-1} \\ \sigma_1,\ldots,\sigma_{k-1}}} b^{i_1\ldots i_{k-1} i}_{\sigma_1\ldots\sigma_{k-1}} \frac{\partial^{k-1}}{\partial p^{i_1}_{\sigma_1} \ldots \partial p^{i_{k-1}}_{\sigma_{k-1}}},$$

and

$$\Delta \mid F_r(\pi) = \sum_{\substack{|\sigma_j| \leqslant r \\ j=1,2,\ldots,k}} \sum_{i_1,\ldots,i_k} a^{i_1\ldots i_k}_{\sigma_1\ldots\sigma_k} \frac{\partial^k}{\partial p^{i_1}_{\sigma_1} \ldots \partial p^{i_k}_{\sigma_k}}$$

Remark 3 of n. 2.4 implies that coefficients $a^{i_1\ldots i_k}_{\sigma_1\ldots\sigma_k}$, $|\sigma_j| \leqslant r$ $j = 1, 2, \ldots, k$ are uniquely defined by the coefficients of generating operators $b^{i_1\ldots i_{k-1} i}_{\sigma_1\ldots\sigma_{k-1}}$, $|\sigma_j| \leqslant 2r$, $j = 1, 2, \ldots, k-1$. For coincidence of two operators from $\ni_k(¶)$ on $F_r(¶)$ it suffices that their generating operators coincide on $F_{2r}(¶)$.

Let $\Delta_1 = \ni_{\nabla_l}$, where $\nabla_l = (\nabla_{l,1}; \ldots, \nabla_{l,m})$ and

$$\nabla_{l,i} = \sum_{\substack{i_1,\ldots,i_{k-2} \\ \sigma_1,\ldots,\sigma_{k-2}}} X^{i_1\ldots i_{k-2} i}_{l\ \sigma_1\ldots\sigma_{k-2}} \frac{\partial^{k-2}}{\partial p^{i_1}_{\sigma_1} \ldots \partial p^{i_{k-2}}_{\sigma_{k-2}}} ; \mathscr{S}_l = (\mathscr{S}_{l,1}, \ldots, \mathscr{S}_{l,m}), l = 1, 2, \ldots, L.$$

Then

$$\Delta \mid F_r(¶) = \operatorname{smbl} \sum_{l=1}^{L} (\ni_{\mathscr{S}_l} \circ \Delta_l) \mid F_r(¶),$$

if coefficients of operators Δ_l and functions $\mathscr{S}_l$ satisfy equations

$$\sum_{l=1}^{L} D_{\sigma_1}(\mathscr{S}_{l,i_1}) X^{i_1\ldots i_{k-1} i}_{l\ \sigma_2\ldots\sigma_{k-1}} = b^{i_1\ldots i_{k-1} i}_{\sigma_1\ldots\sigma_{k-1}}, \tag{4.2.1}$$

$$|\sigma_1|, \ldots, |\sigma_{k-1}| \leqslant 2r; \quad i_1, \ldots, i_{k-1}, \quad i = 1, 2, \ldots, m.$$

Then we rewrite (4.2.1) in the following form

$$\sum_{l=1}^{L} \lambda_l^{(i_1,\sigma_1)} \cdot X_l = b^{(i_1,\sigma_1)} \tag{4.2.2}$$

where $\lambda_l^{(i_1,\sigma_1)} = D_{\sigma_1}(\mathscr{S}_{l,i_1})$, $X_l = X^{i_2\ldots i_{k-1} i}_{l\ \sigma_2\ldots\sigma_{k-1}}$, $b^{(i_1,\sigma_1)} = b^{i_1 i_2\ldots i_{k-1} i}_{\sigma_1\sigma_2\ldots\sigma_{k-1}}$. Let us consider the system (4.2.2) as a system of linear algebraic equations with respect to the unknows X_l. Choose a number L and functions $\mathscr{S}_{l,i_1}$ so that the matrix of (4.2.2) were square and $\det(\lambda_l^{(i_1,\sigma_1)}) \neq 0$ (Then there exist solutions X_l for any right hand side $b^{(i_1,\sigma_1)}$). For this purpose we order lexicographically the set of multi-indices (i_1, σ_1), $i_1 = 1, 2, \ldots, m$, $|\sigma_1| \leqslant 2r$ and assume that it is a range of values of the index l. Rearrange the equations in (4.2.2) in lexicographic order of the free terms $b^{(i_1,\sigma_1)}$. Arrange the summands in the left hand sides are according to the lexicographic order of the unknowns $X_{(i,\sigma)}$, $l = (i, \sigma)$. To verify the condition

$\det(\lambda_{(i,\sigma)}^{(i_1,\sigma_1)}) \neq 0$ choose functions $\lambda_{(i,\sigma)}^{(i_1,\sigma_1)}$ so that the matrix $(\lambda_{(i,\sigma)}^{(i_1,\sigma_1)})$ were lowertriangular with units on the main diagonal (here (i_1, σ_1) is a number of a row, (i, σ) is a number of a column). For this it sufficies to set

$$\mathcal{S}_{(i,\sigma),i} = \begin{cases} q_\sigma, \text{ if } i = i_1, \text{ where } q_\sigma = q_1^{j_1} \cdot \ldots \cdot q_n^{j_n},\ \sigma = (j_1, \ldots, j_n); \\ \\ 0, \text{ if } i \neq i_1 . \end{cases}$$

This proves our statement.

Now, return to the proof of the theorem, which we finish by induction in the order of Δ. The statement of the theorem is trivial for operators $\Delta_1 \in \ni_1(\P)$. Suppose it is true for all operators $\Delta_i \in \ni_i(\P)$, $i < k$. Consider an operator $\Delta_k \in \ni_k(\P)$. On W, it is presentable in the form

$$\Delta_k = \operatorname{smbl} \Delta_k + \Delta_{k-1},$$

where $\Delta_{k-1} \in \ni_{k-1}(\P)$. For any positive integer r there exist operators $\ni_{\mathcal{S}_l} \in \ni_1(\P)$ and $\Box_l \in \ni_{k-1}(\P)$ such that

$$\operatorname{smbl} \Delta_k \,|\, F_r(\P) = \operatorname{smbl}\left(\sum_{l=1}^{L} \ni_{\mathcal{S}_l} \circ \Box_l\right) \Big| F_r(\P).$$

Therefore

$$\Delta_k F_r(\P) = \left(\sum_{l=1}^{L} \ni_{\mathcal{S}_l} \circ \Box_l - \left(\sum_{l=1}^{L} \ni_{\mathcal{S}_l} \circ \Box_l - \right.\right.$$

$$\left.\left. - \operatorname{smbl}\left(\sum_{l=1}^{L} \ni_{\mathcal{S}_l} \circ \Box_l\right)\right) + \Delta_{k-1} \;\Big|\, F_r(\P),\right)$$

where

$$\sum_{l=1}^{L} \ni_{\mathcal{S}_l} \circ \Box_l - \operatorname{smbl}\left(\sum_{l=1}^{L} \ni_{\mathcal{S}_l} \circ \Box_l\right) \in \ni_{k-1}(\P).$$

This formula and inductive hypothesis imply the statement of the theorem for operators from $\ni_k(\P)$. ∎

§5. FREE COEFFICIENTS OF A SECONDARY OPERATOR

5.1. As it was noted in n. 2.4, the coefficients $a_{\sigma_1 \ldots \sigma_s}^{i_1 \ldots i_s}$ of the secondary operator

$\ni_\nabla$ are expressed by recursive formula (2.4.5) in terms of coefficients $a^{i_1 \dots i_{s-1} i}_{\sigma_1 \dots \sigma_{s-1} 0}$ of the generating operator ∇. The latter are independent in general case, i.e. there are no relations between them. Hence to determine the operator $\ni_\nabla$ in general case it is necessary to define all the coefficients of the generating operator.

If coefficients of $\ni_\nabla$ are symmetric, then the expressions of coefficients $a^{i_1 \dots i_r \dots i_s}_{\sigma_1 \dots \sigma_r \dots \sigma_s}$, $r = 1, 2, \dots, s-1$ in terms of coefficients of the generating operator are the relations between the latter ones, because $a^{i_1 \dots i_r \dots i_s}_{\sigma_1 \dots 0 \dots \sigma_s} = a^{i_1 \dots i_{r-1} i_{r+1} \dots i_s i_r}_{\sigma_1 \dots \sigma_{r-1} \sigma_{r+1} \dots \sigma_s 0}$ due to symmetricity. These relations are hint to existence of a family of independent coefficients of the generating operator in terms of which all the other coefficients of ∇, hence all the other coefficients of $\ni_\nabla$ are expressed.

DEFINITION. A family of coefficients of the secondary operator $\ni_\nabla$ is a family of free generators, if the coefficients of this family are independent and all the other coefficients of $\ni_\nabla$ are expressed in terms of them.

5.2. Now, give a constructive method of choosing families of free generators for an operator $\ni_\nabla$ with symmetric coefficients.

First, note that in general case symmetric coefficients of $\ni_\nabla$ satisfy no conditions except (2.4.4). Rewrite these conditions in the following form

$$\sum_{r=1}^{s} a^{i_1 \dots i_r \dots i_s}_{\sigma_1 \dots \sigma_r + 1_j \dots \sigma_s} = D_j(a^{i_1 \dots i_s}_{\sigma_1 \dots \sigma_s}) \tag{5.2.1}$$

for $s = 1, 2, \dots, k; j = 1, 2, \dots, n; i_1, \dots, i_s = 1, 2, \dots, m$; and any $\sigma_1, \dots, \sigma_s$.

Since relations (5.2.1) connect only coefficients of the same homogeneous component $\ni_\nabla$, the search for a family of free generators of $\ni_\nabla$ is reduced to that for each of its homogeneous components. Since each homogeneous component of $\ni_\nabla$ is a secondary operator, in what follows we will seek free generators for an homogeneous secondary operator with symmetric coefficients

$$\ni_\nabla = \sum_{\substack{i_1, \dots, i_k \\ \sigma_1, \dots, \sigma_k}} a^{i_1 \dots i_k}_{\sigma_1 \dots \sigma_k} \frac{\partial^k}{\partial p^{i_1}_{\sigma_1} \dots \partial p^{i_k}_{\sigma_k}} .$$

(The case $k = 1$ is clear).

Since coefficients $a^{i_1 \dots i_k}_{\sigma_1 \dots \sigma_k}$ are symmetric, we will always assume that «two-storeyed» multiindices $\begin{pmatrix} i \\ \sigma \end{pmatrix}$ in these coefficients are arranged in decreasing lexicographic order, i.e. $\begin{pmatrix} i_1 \\ \sigma_1 \end{pmatrix} \geqslant \begin{pmatrix} i_2 \\ \sigma_2 \end{pmatrix} \geqslant \dots \geqslant \begin{pmatrix} i_k \\ \sigma_k \end{pmatrix}$ (Here $\begin{pmatrix} i_r \\ \sigma_r \end{pmatrix} \geqslant \begin{pmatrix} i_{r+1} \\ \sigma_{r+1} \end{pmatrix}$ if and only if $(i_r, i_{r1}, \ldots, i_{rn}) \geqslant (i_{r+1}, i_{r+1,1}, \ldots, i_{r+1,n})$ with respect to the lexicographic order).

The coefficients $a^{i_1 \ldots i_k}_{\sigma_1 \ldots \sigma_k}$ with $|\sigma_1| + \ldots + |\sigma_k| = R$ are refered to as coefficients of level R.

Consider each of the relations (5.2.1) as a linear algebraic equation for coefficients $a^{i_1 \ldots i_k}_{\sigma_1 + 1_j \ldots \sigma_k}, \ldots, a^{i_1 \ldots i_k}_{\sigma_1 \ldots \sigma_k + 1_j}$ of level $R + 1 = |\sigma_1| + \ldots + |\sigma_k| + 1$ generated by coefficeints $a^{i_1 \ldots i_k}_{\sigma_1 \ldots \sigma_k}$ of level R. Thus coefficients of level $R + 1$ are strained by the system (5.2.1) of linear algebraic equations generated by coefficients of level R. No other constraints are imposed on coefficients of level $R + 1$, in general.

Free unknowns of this linear system are called free generators of level $R + 1$.

Further, by induction in R we get that the union of free generators of all levels is a family of free generators for the homogeneous operator $\ni_\nabla$. At the inductive step all the coefficients $a^{i_1 \ldots i_k}_{0 \ldots 0}$ of level 0 are supposed to be free.

Note that free generators of each level may be chosen that they were coefficients of a generating operator.

5.3. Now, illustrate the said in n. 5.2 by the detailed analysis for $n = 1$. In this case the homogeneous operator $\ni_\nabla$ is expressed in the form

$$\ni_\nabla = \sum_{\substack{i_1, \ldots, i_k \\ j_1, \ldots, j_k}} a^{i_1 \ldots i_k}_{j_1 \ldots j_k} \frac{\partial^k}{\partial p^{i_1}_{j_1} \ldots \partial p^{i_k}_{j_k}}, \qquad k \geqslant 2,$$

and relations (5.2.1) are of the form

$$(5.3.1) \qquad \sum_{r=1}^{k} a^{i_1 \ldots i_r \ldots i_k}_{j_1 \ldots j_r + 1 \ldots j_k} = D(a^{i_1 \ldots i_k}_{j_1 \ldots j_k})$$

$$i_1, \ldots, i_k = 1, 2, \ldots, m; \quad j_1, \ldots, j_k = 1, 2, \ldots .$$

Since multiindices $\begin{pmatrix} i_r \\ j_r \end{pmatrix}$, $r = 1, 2, \ldots, k$, are arranged in the decreasing lexicographic order in each coefficient $a^{i_1 \ldots i_k}_{j_1 \ldots j_k}$, then there there is the same set of top indices $i_1 \geqslant i_2 \geqslant \ldots \geqslant i_k$ in any term of (5.3.1). Therefore the system of linear equations (5.3.1) generated by all coefficients of level R splits into subsystems $S^{i_1 \ldots i_k}_R$ each of which connects coefficients of level $R + 1$ with the same set of top indices $i_1 \geqslant \ldots \geqslant i_k$. Hence we must study an arbitrary subsystem of the form $S^{i_1 \ldots i_k}_{R+1}$.

Reduce the system $S^{i_1 \ldots i_k}_{R+1}$ to triangular form. For this arrange the unknowns

$a^{i_1\dots i_k}_{j_1\dots j_k}$ in each equation in accordance with the decreasing lexicographic order of their multiindices $(j_1, \dots, j_k)$, i.e. we put the unknown $a^{i_1\dots i_k}_{j_1\dots j_k}$ to the left of the unknown $a^{i_1\dots i_k}_{l_1\dots l_k}$ if $(j_1, \dots, j_k) > (l_1, \dots, l_k)$ with respect to the lexicographic order, and arrange equations of this system in accordance with the decreasing lexicographic order of multiindices $(j_1, \dots, j_k)$ of free terms $D(a^{i_1\dots i_k}_{j_1\dots j_k})$.

LEMMA 5.3.1. *A system of linear equations $S^{i_1\dots i_k}_{R+1}$ with indicated ordering of equations and the unknowns is of the triangular form.*

Proof. Consider two neighbour equations of the system $S^{i_1\dots i_k}_{R+1}$

$$a^{i_1 i_2\dots i_k}_{j_1+1 j_2\dots j_k} + \dots = D(a^{i_1\dots i_k}_{j_1\dots j_k})$$

and

$$a^{i_1 i_2\dots i_k}_{l_1+1 l_2\dots l_k} + \dots = D(a^{i_1\dots i_k}_{l_1\dots l_k}).$$

Since $(j_1, j_2, \dots, j_k) > (l_1, l_2, \dots, l_k)$, then $(j_1 + 1, j_2, \dots, j_k) > (l_1 + 1, l_2, \dots, l_k)$. ∎

COROLLARY 1. *The main unknowns in the triangular system $S^{i_1\dots i_k}_{R+1}$ are $a^{i_1 i_2\dots i_k}_{j_1 j_2\dots j_k}$ such that*

$$\begin{pmatrix} i_1 \\ j_1 - 1 \end{pmatrix} \geqslant \begin{pmatrix} i_2 \\ j_2 \end{pmatrix}.$$

Proof. Let the coefficient $a^{i_1\dots i_k}_{j_1\dots j_k}$ satisfy $\begin{pmatrix} i_1 \\ j_1 - 1 \end{pmatrix} \geqslant \begin{pmatrix} i_2 \\ j_2 \end{pmatrix}$. Then it is the first in the equation

$$a^{i_1 i_2\dots i_k}_{j_1 j_2\dots j_k} + \dots = D(a^{i_1 i_2\dots i_k}_{j_1-1 j_2\dots j_k})$$

of the triangular system $S^{i_1\dots i_k}_{R+1}$, i.e., it is a main unknown. And vice versa. ∎

COROLLARY 2. *Free unknowns of the triangular system $S^{i_1\dots i_k}_{R+1}$ are the unknowns $a^{i_1\dots i_k}_{j_1\dots j_k}$, which satisfy one of the following conditions:*

1) $i_1 > i_2$ *and* $j_1 = 0$;
2) $i_1 = i_2$ *and* $j_1 = j_2$.

Proof. $\begin{pmatrix} i_1 \\ j_1 - 1 \end{pmatrix} \geqslant \begin{pmatrix} i_2 \\ j_2 \end{pmatrix}$ fails if and only if either $i_1 > i_2$ and $j_1 = 0$ or $i_1 = i_2$ and $j_1 = j_2$. ∎

Thus we get

THEOREM 5.3.1. *Let* $n = 1$ *and* $\ni_\nabla$ *be a homogeneous secondary operator with symmetric coefficients. Then the family of its coefficients* $a^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_k}$ *is a family of free generators if they satisfy one of the following conditions:*

1) $i_1 > i_2$ *and* $j_1 = 0$;
2) $i_1 = i_2$ *and* $j_1 = j_2$.

Now, choose a family of free generators of $\ni_\nabla$ so that its elements were coefficients of the generating operator ∇. For this denote by A a family of free unknowns of $S^{i_1 \ldots i_k}_{R+1}$ obtianed in Corollary 2 of Lemma 5.3.1. Then fix a positive integer s and consider elements $a^{i_1 \ldots i_k}_{j_1 \ldots j_k} \in A$ with $j_k > s$. Transfer $j_k - s$ units from the index j_k of each of these elements in an arbitrary way into $k-1$ first indices $j_1, j_2, \ldots, j_{k-1}$ of this element. Denote by A_s the set of elements obtained in this way.

LEMMA 5.3.2. *For any* s *the elements of* A_s *are expressed in terms of elements of* A_0 *via the equations of the system* $S^{i_1 \ldots i_k}_{R+1}$.

Proof. The lemma is proved by induction in s. The first step of induction, $s = 0$, is trivial. Suppose that elements of the families $A_1, A_2, \ldots, A_s$ are expressed in terms of elements of A_0. Due to (5.3.1)

$$\sum_{r=1}^{k-1} a^{i_1 \ldots i_r \ldots i_{k-1} i_k}_{j_1 \ldots j_r + 1 \ldots j_{k-1} s} + a^{i_1 \ldots i_{k-1} i_k}_{j_1 \ldots j_{k-1} s+1} = D(a^{i_1 \ldots i_{k-1} i_k}_{j_1 \ldots j_{k-1} s}),$$

where $a^{i_1 \ldots i_r \ldots i_{k-1} i_k}_{j_1 \ldots j_r + 1 \ldots j_{k-1} s}$ belong to A_s, $1 \leqslant r \leqslant k-1$ and therefore are expressed in terms of elements of A_0. Hence $a^{i_1 \ldots i_{k-1} i_k}_{j_1 \ldots j_{k-1} s+1}$ is expressed in terms of elements of A_0. ∎

COROLLARY. *Elements of* A *are expressed in terms of elements of* A_0.

LEMMA 5.3.3.

$$A_0 = \{a^{i_1 i_2 \ldots i_{k-1} i_k}_{j_1 + j_k j_2 \ldots j_{k-1} 0} \mid a^{i_1 i_2 \ldots i_{k-1} i_k}_{j_1 j_2 \ldots j_{k-1} j_k} \in A\}.$$

Proof. Let $a^{i_1 \ldots i_{k-1} i_k}_{l_1 \ldots l_{k-1} 0} \in A_0$. If $i_1 > i_2$ then this element is recovered from the element $a^{i_1 i_2 \ldots i_k}_{0 l_2 \ldots l_1} \in A$ by transferring all the units of the last lower index to the first lower index. If $i_1 = i_2$ then this element is recovered from $a^{i_1 i_2 \ldots i_k}_{l_2 l_2 \ldots l_1 - l_2} \in A$

in the same way. ∎

Thus we may take elements of A_0 as free unknowns in the system $S_{R+1}^{i_1 \ldots i_k}$.

COROLLARY. $a_{l_1 l_2 \ldots l_{k-1} 0}^{i_1 i_2 \ldots i_{k-1} i_k} \in A_0$ *if and only if it satisfies one of the following conditions:*

1) $i_1 > i_2$ *and* $\begin{pmatrix} i_{k-1} \\ l_{k-1} \end{pmatrix} \geqslant \begin{pmatrix} i_k \\ l_1 \end{pmatrix}$;
2) $i_1 = i_2$ *and* $\begin{pmatrix} i_{k-1} \\ l_{k-1} \end{pmatrix} \geqslant \begin{pmatrix} i_k \\ l_1 - l_2 \end{pmatrix}$.

Thus we have

THEOREM 5.3.2. *Let $n = 1$ and $\ni_\nabla$ be a homogeneous secondary operator with symmetric coefficients. Then the family of its coefficients $a_{j_1 \ldots j_{k-1} 0}^{i_1 \ldots i_{k-1} i_k}$ satisfying one of the conditions:*

1) $i_1 > i_2$ *and* $\begin{pmatrix} i_{k-1} \\ j_{k-1} \end{pmatrix} \geqslant \begin{pmatrix} i_k \\ j_1 \end{pmatrix}$;
2) $i_1 = i_2$ *and* $\begin{pmatrix} i_{k-1} \\ j_{k-1} \end{pmatrix} \geqslant \begin{pmatrix} i_k \\ j_1 - j_2 \end{pmatrix}$

is a family of free generators.

5.4. We obtained the following results by the same reasoning as in n.n. 5.2 and 5.3.

THEOREM 5.4.1. *Suppose*

$$\ni_\nabla = \sum_{\substack{i_1, \ldots, i_k \\ \sigma_1, \ldots, \sigma_k}} a_{\sigma_1 \ldots \sigma_k}^{i_1 \ldots i_k} \frac{\partial^k}{\partial p_{\sigma_1}^{i_1} \ldots \partial p_{\sigma_k}^{i_k}}$$

is a homogeneous secondary operator with symmetric coefficients. Then

I. *If $n = 2$, $k > 2$, then the family of its coefficients $a_{\sigma_1 \ldots \sigma_k}^{i_1 \ldots i_k}$ satisfying one of the following conditions:*

a) $i_1 > i_2$, $\sigma_1 = (0, 0)$;
b) $i_1 = i_2$, $\sigma_1 = (i + 1, 0)$, $\sigma_2 = (i, 2j + 1)$, *where* $i, j = 0, 1, 2, \ldots$;
c) $i_1 = i_2$, $\sigma_1 = (i + 1, 0)$; $\sigma_2 = (i, 2j)$, $\sigma_3 > (i, j)$, *where* $i, j = 0, 1, 2, \ldots$;
d) $i_1 = i_2$, $\sigma_1 = \sigma_2$

is a family of free generators.

II. *If* $n > 2$ *and* $k = 2$, *then the family of coefficients* $a^{i_1 i_2}_{\sigma_1 \sigma_2}$ *of* $\ni_\nabla$ *satisfying one of the following conditions:*

a) $i_1 > i_2$, $\sigma_1 = (0, 0, \dots, 0)$;

b) $i_1 = i_2$, $\sigma_1 = (\tau, i + 1, 0, \dots, 0)$, $\sigma_2 = (\tau, i, \delta)$, *where* τ, δ *are multiindices and* $|\delta|$ *is odd;*

c) $i_1 = i_2$, $\sigma_1 = \sigma_2$

is a family of free generators.

In general, the direct analysis of the system of linear equations connecting coefficients of the same level is very difficult and cumbersome. We intend to return elsewhere to the problem of finding free generators of secondary operators starting from another arguments.

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